

COHOMOLOGY CLASSES OF STRATA OF DIFFERENTIALS

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ABSTRACT. We introduce a space of stable meromorphic differentials with poles of prescribed orders and define its tautological cohomology ring. This space, just as the space of holomorphic differentials, is stratified according to the set of multiplicities of zeros of the differential. The main goal of this paper is to compute the Poincaré-dual cohomology classes of all strata. We prove that all these classes are tautological and give an algorithm to compute them.

In a second part of the paper we study the Picard group of the strata. We use the tools introduced in the first part to deduce several relations in these Picard groups.

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1. INTRODUCTION

1.1. Stratification of the Hodge Bundle. Let $g \geq 1$. Let \mathcal{M}_g be the space of smooth curves of genus g . The *Hodge bundle*,

$$\mathcal{H}_g \rightarrow \mathcal{M}_g$$

is the vector bundle whose fiber over a point $[C]$ of \mathcal{M}_g is the space of holomorphic differentials on C . A point of \mathcal{H}_g is then a pair $([C], \alpha)$, where C is a curve and α a differential on C . We will denote by $\mathbb{P}\mathcal{H}_g \rightarrow \mathcal{M}_g$ the projectivization of the Hodge bundle.

Notation 1.1. Let Z (for zeros) be a vector (k_1, k_2, \dots, k_n) of positive integers satisfying

$$\sum_{i=1}^n k_i = 2g - 2.$$

We will denote by $\mathbb{P}\mathcal{H}_g(Z)$ the subspace of $\mathbb{P}\mathcal{H}_g$ composed of pairs $([C], \alpha)$ such that α is a differential (defined up to a multiplicative constant) with zeros of orders k_1, \dots, k_n .

The locus $\mathbb{P}\mathcal{H}_g(Z)$ is a smooth orbifold (or a Deligne-Mumford stack), see for instance, [25]. However, neither $\mathbb{P}\mathcal{H}_g$, nor the strata $\mathbb{P}\mathcal{H}_g(Z)$ are compact.

The Hodge bundle has a natural extension to the space of stable curves:

$$\overline{\mathcal{H}}_g \rightarrow \overline{\mathcal{M}}_g.$$

We recall that abelian differentials over a nodal curve are allowed to have simple poles at the nodes with opposite residues on the two branches.

The space $\mathbb{P}\overline{\mathcal{H}}_g$ is compact and smooth, and we can consider the closures $\mathbb{P}\overline{\mathcal{H}}_g(Z)$ of the strata inside this space. Computing the Poincaré-dual cohomology classes of these strata is our motivating problem. In this paper we solve this problem and present a more general computation in the case of meromorphic differentials.

1.2. Stable differentials.

Definition 1.2. Let $n, m \in \mathbb{N}$ and let P (for poles) be a vector (p_1, p_2, \dots, p_m) of positive integers. The space of *stable differentials* $\overline{\mathcal{H}}_{g,n,P}$ is the moduli space of objects $([C], x_1, \dots, x_{n+m}, \alpha)$ where C is a pre-stable curve with $n + m$ marked points and α is a non-zero meromorphic differential on C , such that

- the differential α has no poles outside the m last marked points and nodes;

- the poles at the nodes are at most simple and have opposite residues on the two branches;
- if $p_i > 1$ then the pole at the marked point x_{n+i} is of order exactly p_i ; if $p_i = 1$ then x_i can be a simple pole, a regular point, or a zero of any order;
- the group of isomorphisms of C preserving α and the marked points is finite.

Proposition 1.3. *The forgetful map $\overline{\mathcal{H}}_{g,n,P} \rightarrow \overline{\mathcal{M}}_{g,n+m}$ is an orbifold cone of rank $g - 1 + m + \sum p_i$. Its Segre class is given by*

$$\prod_{i=1}^m \frac{(p_i - 1)^{p_i - 1}}{(p_i - 1)!} \cdot \frac{1 - \lambda_1 + \dots + (-1)^g \lambda_g}{\prod_{i=1}^m (1 - (p_i - 1)\psi_i)}.$$

This proposition will be proved in Section 2.

Notation 1.4. Let P be a vector of positive integers of length m . Let Z be a vector of nonnegative integers of length n . We denote by $A_{g,Z,P} \subset \overline{\mathcal{H}}_{g,n,P}$, the locus of stable differentials $([C], x_1, \dots, x_{n+m}, \alpha)$ such that C is smooth and α has zeros exactly of orders prescribed by Z at the first n marked points. The locus $A_{g,Z,P}$ is invariant under the \mathbb{C}^* -action. We denote by $\mathbb{P}A_{g,Z,P}$ the projectivization of $A_{g,Z,P}$. Moreover, we denote by $\overline{A}_{g,Z,P}$ (respectively $\mathbb{P}\overline{A}_{g,Z,P}$) the closures of $A_{g,Z,P}$ (resp. $\mathbb{P}A_{g,Z,P}$) in the space $\overline{\mathcal{H}}_{g,n,P}$ (respectively in $\mathbb{P}\overline{\mathcal{H}}_{g,n,P}$).

Let V be a vector, in this article we will denote by $|V|$ the sum of elements of V and by $\ell(V)$ the length of V .

Definition 1.5. Given g and P , we will say that Z is *complete* if it satisfies $|Z| - |P| = 2g - 2$. If Z is complete, we will denote by $Z - P$ the vector $(k_1, \dots, k_n, -p_1, \dots, -p_m)$.

1.3. The tautological ring of $\overline{\mathcal{M}}_{g,n}$. Let g and n be nonnegative integers satisfying $2g - 2 + n > 0$. Let $\overline{\mathcal{M}}_{g,n}$ be the space of stable curves of genus g with n marked points. Define the following cohomology classes:

- $\psi_i = c_1(\mathcal{L}_i) \in H^2(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$, where \mathcal{L}_i is the cotangent line bundle at the i^{th} marked point,
- $\kappa_m = \pi_*(\psi_{n+1}^{m+1}) \in H^{2m}(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$, where $\pi : \overline{\mathcal{M}}_{g,n+1} \rightarrow \overline{\mathcal{M}}_{g,n}$ is the forgetful map,
- $\lambda_k = c_k(\overline{\mathcal{H}}_{g,n}) \in H^{2k}(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$, for $k = 1, \dots, g$.

Definition 1.6. A *stable graph* is the datum of

$$\Gamma = (V, H, g : V \rightarrow \mathbb{N}, a : H \rightarrow V, i : H \rightarrow H, E, L)$$

satisfying the following properties:

- V is a vertex set with a genus function g ;
- H is a half-edge set equipped with a vertex assignment a and an involution i ;
- E , the edge set, is defined as the set of length 2 orbits of i in H (self-edges at vertices are permitted);
- (V, E) define a connected graph;
- L is the set of fixed points of i called *legs*;
- for each vertex v , the stability condition holds: $2g(v) - 2 + n(v) > 0$, where $n(v) = \#(a^{-1}(v))$ (the cardinal of $a^{-1}(v)$).

The genus of Γ is defined by $\sum g(v) + \#(E) - \#(V) + 1$.

Let $v(\Gamma)$, $e(\Gamma)$, and $n(\Gamma)$ denote the cardinalities of V , E , and L , respectively. A boundary stratum of the moduli space of curves naturally determines a stable graph of genus g with n legs by considering the dual graph of a generic pointed curve parameterized by the stratum. Thus the boundary strata of $\overline{\mathcal{M}}_{g,n}$ are in 1-to-1 correspondence with stable graphs.

Let Γ be a stable graph. Define the moduli space $\overline{\mathcal{M}}_\Gamma$ by the product

$$\overline{\mathcal{M}}_\Gamma = \prod_{v \in V} \overline{\mathcal{M}}_{g(v), n(v)},$$

and let $\zeta_\Gamma : \overline{\mathcal{M}}_\Gamma \rightarrow \overline{\mathcal{M}}_{g,n}$ be the natural morphism.

Definition 1.7. A *tautological class* is a linear combination of classes β of the form

$$\beta = \zeta_{\Gamma*} \left(\prod_{v \in V} P_v \right),$$

where Γ is a stable graph and P_v is a polynomial in κ , λ and ψ classes on $\overline{\mathcal{M}}_{g(v), n(v)}$.

Proposition 1.8. *The tautological classes form a subring of $H^*(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$.*

See [15] for the description of the product of tautological classes.

Definition 1.9. This subring is called the *tautological ring* of $\overline{\mathcal{M}}_{g,n}$ and denoted by $RH^*(\overline{\mathcal{M}}_{g,n})$.

Remark 1.10. Actually the classes α as above that do not involve λ -classes span the tautological ring. However it will be more convenient for us to use this larger set of generators.

1.4. The tautological ring of $\mathbb{P}\overline{\mathcal{H}}_{g,n,P}$. Let P be a vector of positive integers. From now on, unless specified otherwise, we will denote by $\pi : \overline{\mathcal{M}}_{g,n+1} \rightarrow \overline{\mathcal{M}}_{g,n}$ the forgetful map and by $p : \overline{\mathcal{H}}_{g,n,P} \rightarrow \overline{\mathcal{M}}_{g,n+m}$ the projection from the space of differentials to $\overline{\mathcal{M}}_{g,n}$. Moreover we will use the same notation $p : \mathbb{P}\overline{\mathcal{H}}_{g,n,P} \rightarrow \overline{\mathcal{M}}_{g,n+m}$ for the projectivized cone. Let

$$\mathcal{L} = \mathcal{O}(1) \rightarrow \mathbb{P}\overline{\mathcal{H}}_{g,n,P}$$

be the dual of the tautological line bundle of $\mathbb{P}\overline{\mathcal{H}}_{g,n,P}$, and let

$$\xi = c_1(\mathcal{L}).$$

Definition 1.11. The *tautological ring of $\mathbb{P}\overline{\mathcal{H}}_{g,n,P}$* is the subring of the cohomology ring $H^*(\mathbb{P}\overline{\mathcal{H}}_{g,n,P}, \mathbb{Q})$ generated by ξ and the pull-back of $RH^*(\overline{\mathcal{M}}_{g,n+m})$ under p . We denote it by $RH^*(\overline{\mathcal{H}}_{g,n,P})$.

Remark 1.12. We have $\xi^d = 0$ for $d > \dim(\mathbb{P}\overline{\mathcal{H}}_{g,n,P})$. Therefore the tautological ring of stable differentials is a finite extension of the tautological ring of stable curves. In particular, in absence of poles, the Hodge bundle is a vector bundle and we have

$$RH^*(\mathbb{P}\overline{\mathcal{H}}_{g,n}) = RH^*(\overline{\mathcal{M}}_{g,n})[\xi]/(\xi^g + \lambda_1 \xi^{g-1} + \dots + \lambda_g).$$

1.5. Statement of the results. We have now all elements to state the main theorems of this article.

Theorem 1. *For any vectors Z and P , the class $[\mathbb{P}\overline{A}_{g,Z,P}]$ introduced in Notation 1.4, lies in the tautological ring of $\mathbb{P}\overline{\mathcal{H}}_{g,n,P}$ and is explicitly computable.*

The main ingredient to prove this theorem will be the induction formula of Theorem 5.

Restricting ourselves to the holomorphic case and applying the forgetful map of the marked points we obtain the following corollary.

Theorem 2. *For any vector Z , the class $[\mathbb{P}\overline{\mathcal{H}}_g(Z)]$ introduced in Notation 1.1 lies in the tautological ring of $\mathbb{P}\overline{\mathcal{H}}_g$ and is explicitly computable.*

Remark 1.13. As a guideline for the reader, it will be important to understand that the holomorphic case in Theorem 1 cannot be proved without using strictly meromorphic differentials. Thus Theorem 2 is a consequence of a specific case of Theorem 1 but one cannot avoid to prove Theorem 1 in its full generality.

The second important corollary is obtained by forgetting the differential instead of the marked points. Let $P = (p_1, \dots, p_m)$ be a vector of poles and $Z = (k_1, \dots, k_n)$ be a complete vector of zeros. We define $\mathcal{H}_g(Z - P) \subset \mathcal{M}_{g,n+m}$ as the locus of points (C, x_1, \dots, x_n) that satisfy:

$$\omega_C \left(- \sum_{i=1}^n k_i x_i + \sum_{j=1}^m p_j x_{n+j} \right) \simeq \mathcal{O}_C.$$

We denote by $\overline{\mathcal{H}}_g(Z - P)$ the closure of $\mathcal{H}_g(Z - P)$ in $\overline{\mathcal{M}}_{g,n+m}$.

Theorem 3. *For any vectors Z and P , the class $[\overline{\mathcal{H}}_g(Z - P)]$ lies in the tautological ring of $\overline{\mathcal{M}}_{g,n+m}$ and is explicitly computable.*

Remark 1.14. Theorems 1, 2 and 3 are stated for the Poincaré-dual rational cohomology classes. However, all the results of this paper remain valid in the Chow groups.

In a second part of the Paper (Section 5) we will consider the rational Picard group of the space $\overline{\mathcal{H}}_g(Z - P)$. We will define several natural classes in this Picard group and apply the tools developed in the first part of the paper to deduce a series of relations between these classes: see Theorem 6.

1.6. An example. Here we illustrate the general method used in this article by computing the class of differentials with a double zero $[\mathbb{P}\overline{\mathcal{H}}_g(2, 1, \dots, 1)]$. This computation was carried out by D. Zvonkine in an unpublished note [27] and was the starting point of the present work.

We begin by marking a point, i.e. we study the space $\mathbb{P}\overline{\mathcal{H}}_{g,1}$ of triples $(C, x_1, [\alpha])$ composed of a stable curve C with one marked point x_1 and an abelian differential α modulo a multiplicative constant. Recall that $\mathbb{P}\overline{A}_{g,(2)} \subset \mathbb{P}\overline{\mathcal{H}}_{g,1}$ is the closure of the locus of smooth curves with a double zero at the marked point. In order to compute $[\mathbb{P}\overline{A}_{g,(2)}]$, we consider the line bundle

$$\mathcal{L} \otimes \mathcal{L}_1 \simeq \text{Hom}(\mathcal{L}^\vee, \mathcal{L}_1)$$

over $\mathbb{P}\overline{\mathcal{H}}_{g,1}$. (Recall that \mathcal{L}^\vee is the tautological line bundle of the projectivization $\mathbb{P}\overline{\mathcal{H}}_{g,1}$ and \mathcal{L}_1 is the cotangent line bundle at the marked point x_1 .) We construct a natural section s_1 of this line bundle,

$$\begin{aligned} s_1 : \mathcal{L}^\vee &\rightarrow \mathcal{L}_1 \\ \alpha &\mapsto \alpha(x_1). \end{aligned}$$

Namely, an element of \mathcal{L}^\vee is an abelian differential on C , and we take its restriction to the marked point.

The section s_1 vanishes if and only if the marked point is a zero of the abelian differential. Thus we have the following identity in $H^2(\mathbb{P}\overline{\mathcal{H}}_{g,1})$:

$$[\mathbb{P}\overline{A}_{g,(1)}] = [\{s_1 = 0\}] = c_1(\mathcal{L} \otimes \mathcal{L}_1) = \xi + \psi_1.$$

Now we restrict ourselves to the locus $\{s_1 = 0\}$ and consider the line bundle

$$\mathcal{L} \otimes \mathcal{L}_1^{\otimes 2}.$$

We build a section s_2 of this new line bundle. An element of $\mathcal{L}_{|\{s_1=0\}}^\vee$ is an abelian differential with at least a simple zero at the marked point x_1 . Its next leading term at the marked point is an element of $\mathcal{L}_1^{\otimes 2}$ (we can verify this assertion using a local coordinate at x_1).

As before, s_2 is equal to zero if and only if the marked point is at least a double zero of the abelian differential. However, $\{s_2 = 0\}$ is composed of three components:

- $\mathbb{P}\overline{A}_{g,(2)}$;
- the locus α_e where the marked point lies on an elliptic component attached to the rest of the stable curve at exactly one point and the abelian differential vanishes identically on the elliptic component;
- the locus α_r where the marked point lies on a “rational bridge”, that is, a rational component attached to two components of the stable curve that are not connected except by this rational component (in this case the abelian differential automatically vanishes on the rational bridge).

We deduce the following formula for $[\mathbb{P}\overline{A}_{g,(2)}]$:

$$\begin{aligned} [\mathbb{P}\overline{A}_{g,(2)}] &= [\{s_2 = 0\}] - [\alpha_e] - [\alpha_r] \\ &= (\xi + \psi_1)(\xi + 2\psi_1) - [\alpha_e] - [\alpha_r] \end{aligned}$$

Remark 1.15. We make a series of remarks on this result.

- To transform the above considerations into an actual proof we need to check that the vanishing multiplicity of s_2 along all three components equals 1. We will prove this assertion and its generalization in Section 3.
- Denote by $\pi : \overline{\mathcal{M}}_{g,1} \rightarrow \overline{\mathcal{M}}_g$ the forgetful map then we have, by δ_{sep} the boundary divisor of composed of curves with a separating node, and δ_{nonsep} the boundary divisor of curves with a separating node. Applying the push-forward of π to the above expression of $[\mathbb{P}\overline{A}_{g,(2)}]$ we get

$$[\mathbb{P}\overline{\mathcal{H}}(2, 1, \dots, 1)] = \pi_*[\mathbb{P}\overline{A}_{g,(2)}] = (3\kappa_0\xi + 2\kappa_1) - \delta_{\text{sep}} - 0,$$

The terms $\pi_*(\xi^2)$ and $\pi_*(\alpha_e)$ vanish by the projection formula. Using the relations $\kappa_0 = 2g - 2$, and $\kappa_1 = 12\lambda_1 - \delta_{\text{sep}} - \delta_{\text{nonsep}}$ on $\overline{\mathcal{M}}_g$ (see, for example, [1], chapter 17), we have

$$[\mathbb{P}\overline{\mathcal{H}}(2, 1, \dots, 1)] = (6g - 6)\xi + 24\lambda_1 - 3\delta_{\text{sep}} - 2\delta_{\text{nonsep}}.$$

This formula was first proved by Korotkin and Zograf in 2011 using an analysis of the Bergman tau function [20]. Dawei Chen gave another proof of this result in 2013 using test curves [6].

- In general, to prove Theorem 1 we will work by induction. Let $Z = (k_1, k_2, \dots, k_n)$ and P be vectors. Let $Z' = (k_1, \dots, k_i + 1, \dots, k_n)$. Then we will show that

$$[\mathbb{P}\overline{A}_{g,Z',P}] = (\xi + (k_i + 1)\psi_i) [\mathbb{P}\overline{A}_{g,Z,P}] - \text{boundary terms.}$$

The computation of these boundary terms is the crucial part of the proof.

1.7. Applications and related work.

Classes in the Picard group of $\overline{\mathcal{M}}_g$. Scott Mullane and Dawei Chen gave a closed formula for the class of $\pi_* [\mathbb{P}\overline{\mathcal{H}}_g(Z)]$ in the rational Picard group of $\overline{\mathcal{M}}_g$ for all Z of length $g - 2$ (see [5] and [21]). They used test curves and linear series to compute this formula. This result has the advantage of giving explicit expressions, however it has the drawback of not keeping track of the positions of the zeros and of being restricted to the vectors Z of length $g - 2$ (see Section 4.2 for an example of computation).

Incidence variety compactification. The problem of the compactification of the strata is extensively studied from different approaches in a joint work of Bainbridge, Chen, Gendron, Grushevsky, and Moeller (see [2] and [14]). Their compactification (called *incidence variety compactification*) is slightly different from the one that we use here. We will recall their definitions in Section 3.2 since we will make use of some of their results.

Moduli space of twisted canonical divisors and Double Ramification Cycle. In [12], Farkas and Pandharipande proposed another compactification of the strata. Let g, n such that $2g - 2 + n > 0$. Let $\mu = (k_1, \dots, k_n)$ be a partition of $2g - 2$. We recall that $\mathcal{H}_g(\mu) \subset \mathcal{M}_{g,n}$ is the locus of smooth curves such that $\omega_C(-k_1x_1 - \dots - k_nx_n) \simeq \mathcal{O}_C$ and that we denote by $\overline{\mathcal{H}}_g(\mu)$ its closure in $\overline{\mathcal{M}}_{g,n}$. In [12], Farkas and Pandharipande defined the space of twisted canonical divisors denoted by $\tilde{\mathcal{H}}(\mu)$. The space of twisted canonical divisors is a singular closed subspace of $\overline{\mathcal{M}}_{g,n}$ such that $\overline{\mathcal{H}}(\mu)$ is one of the irreducible components of $\tilde{\mathcal{H}}(\mu)$.

We assume that μ contains at least one negative value. In the appendix of [12], Farkas and Pandharipande defined a class $H_g(\mu)$ in $A_g(\overline{\mathcal{M}}_{g,n})$ (or $H^{2g}(\overline{\mathcal{M}}_{g,n})$): this class is a weighted sum over the classes of irreducible components.

Conjectural expression of $H_g(\mu)$ and Double Ramification Cycle. Let r be a positive integer and (C, x_1, \dots, x_n) be a smooth curve with markings. A r -spin structure is a line bundle L such that $L^{\otimes r} \simeq \omega_C(-k_1x_1 - \dots - k_nx_n)$. We denote the moduli space of r -spin structures by $\mathcal{M}_{g,\mu}^{1/r}$. This space admits a standard compactification by twisted r -spin structures: $\overline{\mathcal{M}}_{g,\mu}^{1/r}$. We denote by $\pi : \overline{\mathcal{C}}_{g,\mu}^{1/r} \rightarrow \overline{\mathcal{M}}_{g,\mu}^{1/r}$ the universal curves and by $\mathbb{L} \rightarrow \overline{\mathcal{C}}_{g,\mu}^{1/r}$ the universal line bundle. The moduli space of twisted r -spin structures has a natural forgetful map $\epsilon : \overline{\mathcal{M}}_{g,\mu}^{1/r} \rightarrow \overline{\mathcal{M}}_{g,n}$; the map ϵ is finite of degree r^{2g-1} . We consider $R\pi_*(\mathbb{L})$ the image of \mathbb{L} in the derived category of

$\overline{\mathcal{M}}_{g,\mu}^{1/r}$. The following diagram sums up the notation:

$$\begin{array}{ccc}
 \mathbb{L} & & \\
 \downarrow & \searrow & \\
 R\pi_*(\mathbb{L}) & & \overline{\mathcal{C}}_{g,n}^{1/r,\mu} \\
 & & \downarrow \pi \\
 & & \overline{\mathcal{M}}_{g,n}^{1/r,\mu} \xrightarrow{\epsilon} \overline{\mathcal{M}}_{g,n}
 \end{array}$$

If μ has at least one negative value, we consider class $c_g^r(\mu) \stackrel{\text{def}}{=} R\pi_*(\mathbb{L}) \in A_g(\overline{\mathcal{M}}_{g,\mu}^{1/r})$. If all values of μ are positive we consider a different class, namely Witten's class: $c_W^r \in (\mu)A_{g-1}(\overline{\mathcal{M}}_{g,\mu}^{1/r})$. There are several equivalent definitions of Witten's class, all of which require several technical tools that we will not describe here (see [26], [7] or [4]). Instead, we consider the two following functions:

$$\begin{aligned}
 P_{g,\mu}, P_{g,\mu}^W : \mathbb{N}^* &\rightarrow A_*(\overline{\mathcal{M}}_{g,n}) \\
 r &\mapsto r\epsilon_*(c_g^r(\mu)), r\epsilon_*(c_W^r(\mu)).
 \end{aligned}$$

Both $P_{g,\mu}$ and $P_{g,\mu}^W$ are polynomials for large values of r . This result is due to Aaron Pixton for $P_{g,\mu}$ (see [17]), and Felix Janda for $P_{g,\mu}^W$ (see [24]). We denote by $\tilde{P}_{g,\mu}$ and $\tilde{P}_{g,\mu}^W$ the asymptotic polynomials. The two following conjectures have been proposed:

Conjecture A. (see [12]) *If μ has at least one negative value then the equality $H_g(\mu) = \tilde{P}_{g,\mu}(0)$ holds in $A_g(\overline{\mathcal{M}}_{g,n})$.*

Conjecture B. (see [24]) *If μ has only positive values then the equality $[\overline{H}_g(\mu)] = \tilde{P}_{g,\mu}^W(0)$ holds in $A_{g-1}(\overline{\mathcal{M}}_{g,n})$.*

These two conjectures are the analogous for differentials of the formula for the so-called Double Ramification cycles: a Double Ramification cycle is the push-forward to the moduli space of curves of the virtual fundamental class of a moduli space of rubber maps to \mathbb{P}^1 with prescribed singularities at 0 and ∞ (see [17]). As a consequence of Theorem 3, we know that the classes $H_g(\mu)$ and $[\overline{H}_g(\mu)]$ are tautological and we have an algorithm to check the validity of the conjectures case by case (see Section 4.2 for examples of computations).

Compactification via log-geometry. J  r  my Gu  r   has defined a moduli space of "rubber" differentials using log geometry. He proves that this space is endowed with a perfect obstruction theory. Moreover, if μ has negative values, this moduli space surjects onto the moduli space of twisted canonical divisors and the class $H_g(\mu)$ is the push-forward of the virtual fundamental cycle (see [16]).

If μ has only positive values, Dawei Chen and Qile Chen have also used log geometry to define a compactification of the strata $\mathcal{H}_g(\mu)$ (see [8]). They also proved that this compactification is endowed with a perfect obstruction theory; however the virtual dimension of their compactification is equal to $\dim(\overline{\mathcal{H}}_g(\mu)) - 1$.

Induction formula for singularities in families. The central result of the present work is the induction formula of Section 3. A similar formula has been proved by Kazarian, Lando and Zvonkine for classes of singularities in families of genus 0

stable maps (see [18]). Their formula contains only the genus 0 part of our induction formula.

They give an interpretation of the induction formula in genus 0 as a generalization of the completed cycle formula of Okounkov and Pandharipande (see [23]). For, now it is not clear if this generalized completed cycle formula has an extension to higher genera.

Computation of the Lyapunov exponents of strata. Our computation of cohomology classes of strata in the space of differentials could be useful for the study of the dynamics of flat surfaces. This idea is developed for example in [20] and [5] based on the work of Kontsevich and Zorich [10] (see Section 5.3.3 for a detailed presentation).

1.8. Plan of the paper. In Section 2 we describe the geometry of the cone of stable differentials and compute its Segre class. In Section 3 we describe the boundary of the strata of differentials with prescribed orders of zeros and we establish the formula for the boundary terms in the induction; we use this result to prove Theorems 1, 2 and 3. In Section 4, we give some examples of explicit computations in the projectivized Hodge bundle or in the moduli space of curves. In Section 5, we introduce several classes in the Picard group of the strata and prove several relations between these classes by using the induction formula.

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2. STABLE DIFFERENTIALS

In this section, we study the space of stable differentials. We prove that it is a cone over $\overline{\mathcal{M}}_{g,n}$ and compute the Segre class. We also define stable differentials on disconnected curves.

2.1. The cone of generalized principal parts. We follow here the approach of [9]. Let X be a projective DM stack.

Definition 2.1. An orbifold cone is a finitely generated sheaf of graded $\%_0 X$ -algebras $S = S^0 \oplus S^1 \oplus S^2 \oplus \dots$ such that $S^0 = \%_0 X$.

Remark 2.2. This definition of cone is weaker than the classical definition of Fulton (see [13]) because we do not ask that S be generated by S^1 . However, we can assign to this object an orbifold $\mathcal{C} = \text{Spec}(S)$ and its projectivization $\mathbb{P}\mathcal{C} = \text{Proj}(S)$ which is also an orbifold (see example below). The projectivization $\mathbb{P}\mathcal{C}$ comes with a natural orbifold line bundle $\%_0(1)$, the dual of the tautological line bundle. We

denote $p : \mathbb{P}\mathcal{C} = \text{Proj}(\mathcal{C}) \rightarrow X$ and $\xi = c^1(\mathcal{O}(1))$. Let $\mathcal{C} \rightarrow X$ be a pure-dimensional cone and r the rank of the cone defined as $\dim(\mathcal{C}) - \dim(X)$. The i -th Segre class of \mathcal{C} is defined as

$$s_i = p_*(\xi^{r+i-1}) \in H^{2i}(X, \mathbb{Q}).$$

Example 2.3. Let us consider the graded algebra $\mathbb{C}[x, y, z]$ such that x is an element of weight 2, y is an element of weight 3 and z is an element of weight 1. This graded algebra which is not generated by its degree 1 elements. The associated projectivized cone over a point is the weighted projective space $\mathbb{P}(2, 3)$ which is the quotient of $(\mathbb{C}^3)^*$ by \mathbb{C}^* with the action:

$$\lambda \cdot (x, y, z) = (\lambda^2 x, \lambda^3 y, \lambda z).$$

Definition 2.4. Let p be an integer greater than 1. A *principal part* of order p at a smooth point of a curve is an equivalence class of germs of meromorphic differentials with a pole of order p ; two germs f_1, f_2 are equivalent if $f_1 - f_2$ is a meromorphic differential with at most a simple pole.

First, we parametrize the space of principal parts at a point. Let z be a local coordinate at $0 \in \mathbb{C}$. A principal part at 0 of order p is given by:

$$\left[\left(\frac{u}{z} \right)^{p-1} + a_1 \left(\frac{u}{z} \right)^{p-2} + \dots + a_{p-2} \left(\frac{u}{z} \right) \right] \frac{dz}{z}$$

with $u \neq 0$. However the choice of (u, a_1, \dots, a_{p-2}) is not unique. Indeed there are $p-1$ choices of u given by the $\zeta^\ell \cdot u$ (with $\zeta^\ell = \exp(\frac{2i\pi\ell}{p-1})$, for $0 \leq \ell \leq p-1$). Therefore the coordinates (u, a_1, \dots, a_{p-1}) parametrize a covering of the space of principal parts. The group of automorphisms of the covering is $\mathbb{Z}/(p-1)\mathbb{Z}$. It acts by

$$\zeta \cdot (u, a_1, \dots, a_{p-1}) = (\zeta u, \zeta a_1, \dots, \zeta^{p-1} a_{p-1}).$$

Moreover, the natural action of \mathbb{C}^* on the space of principal parts in coordinates (u, a_1, \dots, a_{p-2}) is given by

$$\lambda \cdot (u, a_1, \dots, a_{p-1}) = (\lambda^{\frac{1}{p}} u, \lambda^{\frac{1}{p}} a_1, \dots, \lambda^{\frac{p-1}{p}} a_{p-1}).$$

This is well defined up to the action of $\mathbb{Z}/(p-1)\mathbb{Z}$. From now on, we will use the nonstandard notation $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$. Set the weight of u to $\frac{1}{p-1}$ and the weights of the a_j to $\frac{j}{p-1}$. The graded algebra

$$S = \mathbb{C}[u, a_1, \dots, a_{p-2}]^{\mathbb{Z}_{p-1}},$$

is called the *algebra of generalized principal parts* and $P = \text{Spec}(S)$ is the *space of generalized principal parts*. We denote by A the subspace defined by $\{u^{p-1} = 0\}$. It is the Cartier divisor obtained as the image of the Weyl divisor $\{u = 0\} \subset \mathbb{C}^{p-1}$ in the quotient of \mathbb{C}^{p-1} by the action of \mathbb{Z}_{p-1} . The space of principal parts embeds into P as the complementary of A .

Lemma 2.5. *A change of local coordinate z induces an isomorphism of S that preserves the grading and acts trivially on the quotient algebra S/I_u , where I_u is the ideal of monomials containing u .*

Proof. Let $z = f(w) = \alpha_1 w + \alpha_2 w^2 + \dots$ be a local coordinates change. We denote by $(u', a'_1, \dots, a'_{p-2})$ the parameters of the presentation of principal parts in coordinate w . We have the transformation:

$$\begin{aligned} u &\mapsto \alpha_1 u \\ a_1 &\mapsto a_1 + \gamma_{1,1} u \\ a_2 &\mapsto a_2 + \gamma_{2,1} u a_1 + \gamma_{2,2} u^2 \\ &\dots \end{aligned}$$

where the $\gamma_{i,j}$ are constants depending only on the order of the principal part. By taking u to be 0, we see that the coordinates (a_1, \dots, a_{p-2}) of A_{p-1} are independent of the choice of local coordinate. \square

Remark 2.6. The space of principal parts of differentials differs from the cone of principal parts of functions of [9] only by the coefficients $\gamma_{i,j}$.

Now, let g, n be nonnegative integers such that $2g - 2 + n > 0$. Let $i \in \llbracket 1, n \rrbracket$ and $p_i \geq 2$. We denote by \mathbb{P}_i the sheaf of graded algebras over $\overline{\mathcal{M}}_{g,n}$, whose stalks at a point (C, x_1, \dots, x_n) is given by \mathbb{Z}_{p_i-1} -invariants polynomials in Laurent coefficients, and we denote by $\mathcal{P}^i = \text{Spec}(\mathbb{P}_i)$ the associated orbifold. In the same way, we define by the cone \mathcal{A}^i of principal parts with vanishing leading coefficients. The latter is a trivial cone over $\overline{\mathcal{M}}_{g,n}$ and we have

Lemma 2.7. *The cone \mathcal{A}^i is the product of $\overline{\mathcal{M}}_{g,n}$ with the weighted projective space with weights $(\frac{1}{p_i-1}, \dots, \frac{p_i-2}{p_i-1})$ quotiented by the action of \mathbb{Z}_{p_i-1} . Moreover the Segre classes of \mathcal{A}^i and \mathcal{P}^i are given by*

$$\begin{aligned} s(\mathcal{A}^i) &= \frac{(p_i - 1)^{p_i-2}}{(p_i - 1)!} \\ s(\mathcal{P}^i) &= \frac{(p_i - 1)^{p_i-1}}{(p_i - 1)!} \cdot \frac{1}{1 - (p_i - 1)\psi_i}. \end{aligned}$$

Proof. The proof is based on the same arguments as for the cone of principal parts of functions. The section u^{p_i-1} is a section of the line bundle $\mathcal{L}_i^{-\otimes(p_i-1)}$ which vanishes with multiplicity $p_i - 1$ along \mathcal{A}^i . \square

2.2. The space of stable differentials. Let g, n, m be nonnegative integers satisfying $2g - 2 + n + m > 0$. Let $P = (p_1, p_2, \dots, p_m)$ be a vector of positive integers. For all $1 \leq i \leq m$, we denote by \mathcal{P}^i the cone of principal parts of order p_i at the $(n + i)^{\text{th}}$ marked point. Let

$$p : \overline{\mathcal{H}}_{g,n,P} \rightarrow \overline{\mathcal{M}}_{g,n+m}$$

be the space of stable differentials of Definition 1.2 together with the forgetful map. The space $\overline{\mathcal{H}}_{g,n,P}$ is not a vector bundle because there is no natural definition of the sum of two differentials with fixed orders of poles. We will prove that $\overline{\mathcal{H}}_{g,n,P}$ is a cone over $\overline{\mathcal{M}}_{g,n+m}$.

The \mathbb{C}^ -invariant locus.* The space $\overline{\mathcal{H}}_{g,n,P}$ comes with a natural \mathbb{C}^* -action. Let us describe the \mathbb{C}^* -invariant locus. Let C be a curve in $\overline{\mathcal{M}}_{g,n+m}$. We define \tilde{C} a pre-stable curve composed of $m' + 1$ components (here m' is the number of elements of P strictly superior to 1): one main component isomorphic to C and m' rational components carrying the poles of order greater than 1. We define $\tilde{\alpha}$ on \tilde{C} as the

differential vanishing on the main component and given by $z^{p_i-1} \frac{dz}{z}$ on the exterior ones. We can easily check that the action of \mathbb{C}^* on $(\tilde{C}, x_1, \dots, x_{n+m}, \tilde{\alpha})$ is trivial. We will denote by $\overline{\mathcal{M}}_{g,n+m}^* \subset \overline{\mathcal{H}}_{g,n,\mathbb{P}}$ the locus of such differentials.

The moduli space $\overline{\mathcal{M}}_{g,n+m}^*$ is not isomorphic to $\overline{\mathcal{M}}_{g,n+m}$ as a stack because the object $(\tilde{C}, x_1, \dots, x_{n+m}, \tilde{\alpha})$ has more automorphisms than the marked curve (C, x_1, \dots, x_{n+m}) . The space $\overline{\mathcal{M}}_{g,n+m}^*$ is equal to the quotient of $\overline{\mathcal{M}}_{g,n}$ by the trivial action of the group $\prod_{i=1}^m \mathbb{Z}_{p_i-1}$. We define the map of stacks $\iota : \overline{\mathcal{M}}_{g,n+m} \rightarrow \overline{\mathcal{H}}_{g,n,P}$ which maps (C, x_1, \dots, x_{n+m}) to $(\tilde{C}, x_1, \dots, x_{n+m}, \tilde{\alpha})$.

The map $\Phi : \overline{\mathcal{H}}_{g,n,P} \rightarrow \bigoplus_{i=1}^m \mathcal{P}^i$. Let $(C, x_1, \dots, x_{n+m}, \alpha)$ be a point of $\overline{\mathcal{H}}_{g,n,P}$ such that C is a stable curve. Then we can define $\Phi_i(\alpha)$ to be the principal part of α at the $(n+i)$ th marked point. The map Φ_i defined this way is equivariant with respect to the \mathbb{C}^* -action.

Proposition 2.8. *The map Φ_i has a natural \mathbb{C}^* -equivariant extension to the locus of stable differentials with unstable underlying curves.*

Proof. For all $1 \leq i \leq m$, there exists a set of local coordinates of the universal curve $\pi : \overline{\mathcal{C}}_{g,n} \rightarrow \overline{\mathcal{M}}_{g,n}$ such that $(X_1, \dots, X_{3g-3+n+m}, z_i)$ are local coordinates of $\overline{\mathcal{M}}_{g,n}$ and z_i is a coordinate of the fibers of π centered at x_{n+i} . In these coordinates a differential with poles exactly of orders $p_i - 1$ at x_{n+i} will be given by

$$\alpha = \left[\left(\frac{u_i}{z_i} \right)^{p_i-1} + a_{i,1} \left(\frac{u_i}{z_i} \right)^{p_i-2} + \dots + a_{i,p_i-1} \right] \frac{dz_i}{z_i} + \text{higher order terms.}$$

For such point, the image of α under the map Φ_i is the point $(u_i, a_{i,1}, \dots, a_{i,p_i-2})$ which belongs to $\mathcal{P}^i \setminus \mathcal{A}^i$. Now the problem is to extend this maps Φ_i when $u_i \rightarrow 0$. For this purpose we add a coordinate w_i such that $z_i w_i = u_i$. In the coordinate w_i the differential α is given by

$$\alpha = - \left[w_i^{p_i-1} + a_{i,1} w_i^{p_i-2} + \dots + a_{i,p_i-1} \right] \frac{dw_i}{w_i} + \text{higher order terms.}$$

When $u_i = 0$, the coordinate w_i is a global coordinate of a rational component attached to curve at $z_i = 0$. On the main component, the differential α is given by $\alpha_m = a_{i,p_i-1} \frac{dz_i}{z_i} + \text{higher order terms}$. On the rational component the differential α_r is given by $- \left[w_i^{p_i-1} + a_{i,1} w_i^{p_i-2} + \dots + a_{i,p_i-1} \right] \frac{dw_i}{w_i}$. Now we define the image of this stable differential by Φ_i to be $(0, a_{i,1}, \dots, a_{i,p_i-2})$. This image belongs to \mathcal{A}_i . This definition of Φ_i does not depend on the choice of the coordinate z_i . \square

We define the map $\Phi : \overline{\mathcal{H}}_{g,n,P} \rightarrow \mathcal{P} \stackrel{\text{def}}{=} \bigoplus_{i=1}^m \mathcal{P}_i$ as the sum $\bigoplus_{i=1}^m \Phi_i$.

Now, we denote by

$$\overline{\mathcal{H}}_{g,n,m} \rightarrow \overline{\mathcal{M}}_{g,n+m}$$

the vector bundle whose fiber over $([C], x_1, \dots, x_{n+m})$ is given by $H^0(C, \omega_C(\sum_{i=1}^m x_{n+i}))$.

The map $\Psi : \overline{\mathcal{H}}_{g,n,m} \rightarrow \overline{\mathcal{H}}_{g,n,P}$. We define the map Ψ as the inclusion of $\overline{\mathcal{H}}_{g,n,m}$ in $\overline{\mathcal{H}}_{g,n,P}$ described as follows. Let $(C, x_1, \dots, x_{n+m}, \alpha)$ be a point in $\overline{\mathcal{H}}_{g,n,m}$. The image of this point by Ψ is given by the curve \tilde{C} constructed as for the \mathbb{C}^* -invariant locus and the differential $\tilde{\alpha}$ is given by α on the main component and $(z^{p_i-1} + \text{res}_{x_{n+i}}(\alpha)) \frac{dz}{z}$ on the exterior rational components.

The space $\overline{\mathcal{H}}_{g,n,m}$ is a vector bundle and the space $\overline{\mathcal{H}}_{g,n,P}$ is the central cone of the exact sequence of cones (see [13] proposition 4.1.6):

$$0 \rightarrow \overline{\mathcal{H}}_{g,n,m} \xrightarrow{\Psi} \overline{\mathcal{H}}_{g,n,P} \xrightarrow{\Phi} \mathcal{P} \rightarrow 0.$$

Indeed the difference of two differentials with the same principal parts is an element of $\overline{\mathcal{H}}_{g,n,m}$. By [13] proposition 4.1.6, the Segre class of the central cone is given by

$$\begin{aligned} s(\overline{\mathcal{H}}_{g,n,P}) &= c(\overline{\mathcal{H}}_{g,n,m})^{-1} \cdot s\left(\bigoplus_{i=n+1}^m \mathcal{P}_i\right) \\ &= \prod_{i=1}^m \frac{(p_i - 1)^{p_i - 1}}{(p_i - 1)!} \cdot \frac{1 - \lambda_1 + \dots + (-1)^g \lambda_g}{\prod_{i=1}^m (1 - (p_i - 1)\psi_i)}. \end{aligned}$$

From the first line to the second we have used the fact that $c(\overline{\mathcal{H}}_g)^{-1} = c(\overline{\mathcal{H}}_g^\vee)$ (see [22]).

2.3. Residues. Let g, n, m and P be as in the previous section. Let \mathcal{R} be the vector subspace of \mathbb{C}^m defined by

$$\mathcal{R} = \{(r_1, r_2, \dots, r_m) / r_1 + r_2 + \dots + r_m = 0\}.$$

The vector space \mathcal{R} will be called the *space of residues*. The following map is equivariant with respect to the \mathbb{C}^* -action:

$$\begin{aligned} \text{res} : \overline{\mathcal{H}}_{g,n,P} &\rightarrow \mathcal{R} \\ \alpha &\mapsto (\text{res}_{x_{n+1}}(\alpha), \text{res}_{x_{n+2}}(\alpha), \dots, \text{res}_{x_{n+m}}(\alpha)). \end{aligned}$$

Let $\overline{\mathcal{H}}_{g,n,P}^0$ be the space of stable differentials with vanishing residues. Recall that $\overline{\mathcal{H}}_{g,n,m}$ is the space of meromorphic differentials on stable curves with at most simple poles at the m last marked points. We have the exact sequence of vector bundles over $\overline{\mathcal{M}}_{g,n+m}$

$$0 \rightarrow \overline{\mathcal{H}}_{g,n+m} \rightarrow \overline{\mathcal{H}}_{g,n,m} \xrightarrow{\text{res}} \mathcal{R} \rightarrow 0,$$

where the first term is the Hodge bundle over $\overline{\mathcal{M}}_{g,n+m}$. Therefore, following the construction of the previous section, we can construct $\overline{\mathcal{H}}_{g,n,P}^0$ as the central cone of the exact sequence:

$$0 \rightarrow \overline{\mathcal{H}}_{g,n+m} \rightarrow \overline{\mathcal{H}}_{g,n,P}^0 \rightarrow \mathcal{P} \rightarrow 0.$$

Thus the space $\overline{\mathcal{H}}_{g,n,P}^0$ is a closed subcone of $\overline{\mathcal{H}}_{g,n,P}$. More generally we can define the following.

Definition 2.9. Let R be a vector subspace of \mathcal{R} . We define $\overline{\mathcal{H}}_{g,n,P}^R$ the space of stable differentials with vector of residues lying in R . We will call R a *space of residue conditions*.

Lemma 2.10. Let $R \subset \mathcal{R}$ be a vector subspace.

- The space $\overline{\mathcal{H}}_{g,n,P}^R$ is a closed subcone of $\overline{\mathcal{H}}_{g,n,P}$ of codimension $m - 1 - \dim(R)$.
- The Segre classes of $\overline{\mathcal{H}}_{g,n,P}^R$ and $\overline{\mathcal{H}}_{g,n,P}$ are equal.

- The Poincaré-dual class of $\mathbb{P}\overline{\mathcal{H}}_{g,n,P}^R$ is given by

$$\left[\mathbb{P}\overline{\mathcal{H}}_{g,n,P}^R\right] = \xi^{m-1-\dim(R)}.$$

Proof. Let $\text{proj} : \mathcal{R} \rightarrow \mathcal{R}/R$ be the projection onto \mathcal{R}/R . We define the space $\overline{\mathcal{H}}_{g,n,m}^R \subset \overline{\mathcal{H}}_{g,n,m}$ to be the space of meromorphic differentials with residues lying in R . We have the exact sequence

$$0 \rightarrow \overline{\mathcal{H}}_{g,n,m}^R \rightarrow \overline{\mathcal{H}}_{g,n,m} \rightarrow \mathcal{R}/R \rightarrow 0,$$

where the last arrow is the composition of the residue map with the projection onto \mathcal{R}/R . The space $\overline{\mathcal{H}}_{g,n,m}^R$ is of codimension $m-1-\dim(R)$ in $\overline{\mathcal{H}}_{g,n,m}$ and this way we construct $\overline{\mathcal{H}}_{g,n,P}^R$ as the central cone in the exact sequence $0 \rightarrow \overline{\mathcal{H}}_{g,n,m}^R \rightarrow \overline{\mathcal{H}}_{g,n,P}^R \rightarrow \mathcal{P} \rightarrow 0$. Therefore, the space $\overline{\mathcal{H}}_{g,n,P}^R$ is a subcone of $\overline{\mathcal{H}}_{g,n,P}$ of codimension $m-1-\dim(R)$.

The Chern classes of $\overline{\mathcal{H}}_{g,n,m}^R$ and $\overline{\mathcal{H}}_{g,n,m}$ are equal because the bundle \mathcal{R}/R is trivial, thus the Segre classes of $\overline{\mathcal{H}}_{g,n,P}^R$ and $\overline{\mathcal{H}}_{g,n,P}$ are also equal.

Now we study the bundle $\%_0(1) \otimes p^*(\mathcal{R}/R) \rightarrow \mathbb{P}\overline{\mathcal{H}}_{g,n,P}$, where we recall that $p : \mathbb{P}\overline{\mathcal{H}}_{g,n,P} \rightarrow \overline{\mathcal{M}}_{g,n+m}$ is the forgetful map. We have $\%_0(1) \otimes p^*(\mathcal{R}/R) \simeq \text{Hom}(\%_0(-1), p^*(\mathcal{R}/R))$. A section of this vector bundle is given by:

$$s : \alpha \mapsto \text{proj} \circ \text{res}(\alpha).$$

The vanishing locus of s is $\mathbb{P}\overline{\mathcal{H}}_{g,n,P}^R$ which is of codimension $m-1-\dim(R)$. Thus the Poincaré-dual class of $\mathbb{P}\overline{\mathcal{H}}_{g,n,P}^R$ in $H^*(\mathbb{P}\overline{\mathcal{H}}_{g,n,P}, \mathbb{Q})$ is given by

$$d \cdot c_{\text{top}}(\%_0(1) \otimes p^*(\mathcal{R}/R)) = d \cdot \xi^{m-1-\dim(R)}$$

where d is a rational number. We will see that the residue map is a submersion (see Corollary 2.15). Therefore the vanishing order of s along R is equal to 1 and the Poincaré-dual class of $\mathbb{P}\overline{\mathcal{H}}_{g,n,P}^R$ is $\xi^{m-1-\dim(R)}$. \square

2.4. Standard coordinates. In this section we describe how to parametrize differentials with prescribed singularities. We use the notation $\Delta_\rho = \{z \in \mathbb{C} : |z| < \rho\}$ for the disks of radius $\rho \in \mathbb{R}^+$ and $A_{\rho_1, \rho_2} = \{z \in \mathbb{C} : \rho_1 < |z| < \rho_2\}$ for the annulus of parameters $0 < \rho_1 < \rho_2$.

Let α be a meromorphic differential on a small disk $\Delta_\rho \subset \mathbb{C}$. We denote by r the residue of α at 0. Then, there exists a conformal map $\varphi : \Delta_{\rho'} \rightarrow \Delta_\rho$ for ρ' small enough, such that: $\varphi(0) = 0$ and

$$\varphi^*(\alpha) = \begin{cases} d(z^k) & \text{if } 0 \text{ is a zero of order } k-1; \\ r \frac{dz}{z} & \text{if } 0 \text{ is a pole of order } 1; \\ d\left(\frac{1}{z^k}\right) + r \frac{dz}{z} & \text{if } 0 \text{ is a pole of order } k+1. \end{cases}$$

The map φ is unique up to multiplication of the coordinate z by a k -th root of unity when 0 is a zero of order $k-1$ or a pole of order $k+1$. The coordinate z will be called the *standard coordinate*.

More generally, if U is an open neighborhood of 0 in \mathbb{C}^n and α_u is a holomorphic family of differentials on Δ_ρ such that the order of α_u at 0 is constant, then there exists an holomorphic map $\varphi : \tilde{U} \times \Delta_{\rho'} \rightarrow \Delta_\rho$ such that $\varphi(u, \cdot)^*(\alpha_u)$ is in the

standard form for some neighborhood of 0, \tilde{U} . Once again the map φ is unique up to multiplication of the standard coordinate by a root of unity.

Now the following classical lemma describes the deformations of $d(z^k)$ (see [19] for a proof):

Lemma 2.11. *Let $\rho > 0$ and $U \subset \mathbb{C}^n$ be a domain containing 0. Let α_u be a family of holomorphic differentials on Δ_ρ such that α_0 has a zero of order k at the origin. Then, there exists $\rho' > 0$, a neighborhood of 0 in \mathbb{C}^{k-2} , \mathcal{Z} and a conformal map*

$$\varphi : U \times \Delta_{\rho'} \rightarrow \Delta_\rho \times \mathcal{Z}$$

such that that $\varphi(u, \cdot)^(\alpha_u) = d(z^k + a_{k-2}z^{k-2} \dots + a_1z)$. The map φ is unique up to multiplication of z by a k -th root of unity.*

Now we would like to generalize the above lemma for deformations with poles of order 1.

Definition 2.12. Let $\rho > 0$ and $U \subset \mathbb{C}^n$ be a domain containing 0. Let α be a differential on Δ_ρ in the standard form $d(z^k)$. A *standard deformation* of α is defined by a holomorphic function $\beta : U \times \Delta_\rho \rightarrow \mathbb{C}$ satisfying $\beta(0, z) = 0$. A standard deformation associated to β is the family of differentials on Δ_ρ parametrized by U

$$\alpha_u = d(z^k) + \frac{\beta(u, z)}{z} dz.$$

In general, there exists no standard coordinate for a standard deformation. However, the following proposition has been proved in [2] (see Theorem 4.3).

Proposition 2.13. *We consider the annulus A_{ρ_1, ρ_2} for any choice of $0 < \rho_1 < \rho_2 < \rho$.*

Chose a point $p \in A_{\rho_1, \rho_2}$ and $\zeta^\ell = \exp(\frac{2i\pi\ell}{k})$ a k -th root of unity. Chose a map $\sigma : U \rightarrow \Delta_\rho$ such that $\sigma(0) = \zeta^\ell p$. Then there exists a neighborhood \tilde{U} of 0 in U and an holomorphic map $\varphi : \tilde{U} \times A_{\rho_1, \rho_2} \rightarrow \Delta_R$ such that

$$\varphi_u^*(\alpha_u) = d(z^k) + \frac{\beta(u, 0)}{z} dz,$$

and $\varphi(0, z) = \zeta^\ell z$ and $\varphi(u, p) = \sigma(u)$ for all $u \in \tilde{U}$ and $z \in A_{\rho_1, \rho_2}$. For U_0 small enough, the map φ is unique.

Let g, n and m be positive integers such that $2g - 2 + n + m > 0$. Let P be a vector of m positive integers and let $R \subset \mathcal{R}$ be a vector subspace. We have described the local parametrization of families of differentials, we will use it to describe how to parametrize the strata $A_{g, Z, P}^R$ and their neighborhood in $\mathbb{P}\overline{\mathcal{H}}_{g, n, P}$.

Lemma 2.14. *There exists a neighborhood V of $A_{g, Z, P}$ in $\overline{\mathcal{H}}_{g, n, P}$ and a holomorphic retraction $\eta : V \rightarrow A_{g, Z, P}$ such that η preserves the residues at the poles.*

Proof. Let Σ be a topological surface of genus g with $n + m$ marked points. Let $(\gamma_1, \dots, \gamma_{2g-2+n+m})$ be a basis of the relative homology group

$$H = H_1(\Sigma \setminus \{x_{n+1}, \dots, x_{n+m}\}, \{x_1, \dots, x_n\}; \mathbb{Z}).$$

The space H is a covering of $A_{g, Z, P}$: given complex numbers $\alpha_1, \dots, \alpha_{2g-2+n+m}$ we can reconstruct a unique complex structure together with a differential α such that $\int_{\gamma_i} \alpha = \alpha_i$.

Let $y_0 = (C, x_1, \dots, x_{n+m}, \alpha)$ be a point of $A_{g,Z,P}$. A neighborhood of y_0 in $A_{g,Z,P}$ is parametrized by an open set U in H . A neighborhood of y_0 in $\overline{\mathcal{H}}_{g,n,P}$ is parametrized by $U \times \prod_{i=1}^n \mathcal{Z}_i$ where \mathcal{Z}_i is a neighborhood of 0 in \mathbb{C}^{k_i} . The $(\alpha_i)_{i=1, \dots, 2g-2+n+m}$ define a unique complex structure and a unique differential. Let z_i be a standard coordinate centered at x_i : i.e. the differential α is given by $d(z_i^{k_i+1})$. The coordinate z_i is unique up to multiplication by a $(k_i + 1)^{\text{st}}$ -root of unity. By Lemma 2.11, deformations of the differential α determines a local coordinate z'_i of the surface Σ centered at x_i and parameters $(a_{i,j})_{j=i, \dots, k_i+1}$ such that the deformed differential is given in the coordinate z'_i by

$$\alpha = d(z_i'^{k_i+1} + a_{i,1} z_i'^{k_i} + a_{i,2} z_i'^{k_i-1} + \dots + a_{i,k_i} z_i').$$

The local coordinate z'_i and the parameters $(a_{i,j})_{j=1, \dots, k_i} \in \prod_{i=1}^n \mathcal{Z}_i$ are unique up to a multiplication of z'_i by a $(k_i + 1)^{\text{st}}$ -root of unity. Now, away from the zeros we can chose a unique change of coordinate $z_i = \varphi(z'_i)$ such that $\varphi'(0) = 1$. The new complex structure is uniquely determined by the choice of the local coordinate z_i and the $a_{i,j}$'s.

The retraction η in this neighborhood of y_0 is defined as the projection along $\prod_{i=1}^n \mathcal{Z}_i$ onto U . This projection is well-defined: indeed given a multiplication of the coordinate z_i by a $(k_i + 1)^{\text{st}}$ -root of unity, we get the same definition of η . Moreover the map η does not depend on the coordinates $(\alpha_i)_{i=1, \dots, 2g-2+n+m}$, thus the definition of η does not depends of the choice of the open set $U \subset H$. Finally, the retraction η does not change the differential away from the zeros; in particular it does not change the residues at the poles. \square

Corollary 2.15. *The residue map restricted to $A_{g,Z,P}$ is a submersion. More generally, if $R \subset \mathcal{R}$ is any vector subspace, the residue map $A_{g,Z,P}^R \rightarrow R$ is a submersion.*

Proof. Let $(C, x_1, \dots, x_{n+m}, \alpha)$ be a point of $A_{g,Z,P}$. Let $\mathbf{r} = (r_1, \dots, r_m)$ be a vector in \mathcal{R} . There exists a meromorphic differential φ on C with at most simple poles at the m last marked points with residues prescribed by \mathbf{r} . Let Δ be a disk of \mathbb{C} centered at 0 and parametrized by ϵ . Let η be the retraction map of Lemma 2.14. The residues of $\eta(\alpha + \epsilon\varphi)$ at the poles are given by

$$\text{res}_{x_{n+i}}(\alpha) + \epsilon r_i.$$

Thus the vector \mathbf{r} belongs to the image of the tangent space of $A_{g,Z,P}$ under the differential of the map res . The same result stands if we restrict the tangent direction to a vector subspace R . \square

Now let $g, n, n', m \geq 0$ such that $2g - 2 + n + n' + m \geq 0$. Let $P = (p_1, \dots, p_m)$ be a vector of positive integer. Let $Z = (k_1, \dots, k_n, k_{n+1}, \dots, k_{n+n'})$ be a vector of nonnegative integers of length $n + n'$. We denote by $P' = (p_1, \dots, p_m, 1, \dots, 1)$ the vector obtained from p by adding n' times 1 and by $Z' = (k_1, \dots, k_n)$ the vector obtained by erasing the last n' entries of Z . The space $\overline{\mathcal{H}}_{g,n+n',P}$ is naturally a closed subspace of $\overline{\mathcal{H}}_{g,n,P'}$. We denote by \mathcal{R} and \mathcal{R}' the vector spaces of residues of $\overline{\mathcal{H}}_{g,n+n',P}$ and $\overline{\mathcal{H}}_{g,n,P'}$. Let R' be a vector subspace of \mathcal{R}' . The vector space \mathcal{R} is a vector subspace of \mathcal{R}' , we will denote by $R = \mathcal{R} \cap R'$. Now we have the natural closed inclusions

$$A_{g,Z,P}^R \subset A_{g,Z',P}^R \subset A_{g,Z',P'}^{R'}.$$

Proposition 2.16. *Let y_0 be a point in $A_{g,Z,P}^R$. Let U be neighborhood of y_0 in $A_{g,Z,P}^R$. There exists a neighborhood V of y_0 in $A_{g,Z',P'}^{R'}$ and a map*

$$\phi : V \xrightarrow{\sim} U \times \left(\prod_{i=1}^{n'} \mathcal{Z}_i \right) \times \mathcal{Z}$$

where:

- \mathcal{Z}_i is a neighborhood of 0 in $\mathbb{C}^{k_{n+i}}$ for all $1 \leq i \leq n'$ and \mathcal{Z} is a neighborhood of 0 in R'/R ;
- if Δ_ρ is a disk and $s : U \times \Delta_\rho \rightarrow (\prod \mathcal{Z}_i) \times \mathcal{Z}$ is a holomorphic map such that $s(u, 0) = 0$ then the family of differentials

$$\begin{aligned} \tilde{s} : U \times \Delta_\rho &\rightarrow V \\ (u, \epsilon) &\mapsto \phi^{-1}(u, s(u, \epsilon)) \end{aligned}$$

is a standard deformation of $d(z^{k_{n+i}+1})$ for all $1 \leq i \leq n'$.

Proof. We have seen that a neighborhood of U in $A_{g,Z',P'}^R$ is isomorphic to $U \times \prod_{i=1}^{n+n'} \mathcal{Z}_i$. For all $1 \leq i \leq n'$, the differential at the marked point x_{n+i} is given by $d(z^{k_{n+i}} + a_1 z^{k_{n+i}+1} + \dots)$ (Lemma 2.11).

Now, we chose a set of meromorphic differentials φ_i with simple poles at the marked points such that the vectors of residues \mathbf{r}_i of φ_i form a basis of R'/R . The residue map $A_{g,Z',P'}^{R'} \rightarrow R'$ is a submersion (Corollary 2.15). Thus a neighborhood of $U \times \prod \mathcal{Z}_i$ in $A_{g,Z',P'}^{R'}$ is naturally identified with a $U \times (\prod \mathcal{Z}_i) \times \mathcal{Z}$ with \mathcal{Z} neighborhood of 0 in R'/R , the identification being given by adding a linear combination of the φ_i 's.

Both the deformations of U into $U \times \prod \mathcal{Z}_i$ and the deformations of $U \times \prod \mathcal{Z}_i$ into $U \times (\prod \mathcal{Z}_i) \times \mathcal{Z}$ are standard deformations at the marked point x_{n+i} for $1 \leq i \leq n'$. \square

The isomorphism ϕ is not unique. It depends of the choice of a standard coordinates at the x_{n+i} for $1 \leq i \leq n'$ and of the choice of the differentials φ_i with simple poles.

Proposition 2.17. *Given such a choice of ϕ , Proposition 2.16 defines a local retraction $\eta : V \rightarrow U$ such that $\eta \circ \tilde{s} = \text{Id}_U$ for any holomorphic section $s : U \times \Delta_\rho \rightarrow (\prod \mathcal{Z}_i) \times \mathcal{Z}$.*

2.5. Dimension of the strata. Let g, n, m be positive integer such that $2g - 2 + n + m > 0$, P a vector of m positive integers, and R a vector subspace of \mathcal{R} . Let Z be a vector of n nonnegative integers. If the context is clear, we will denote by the same letter the map $p : \overline{\mathcal{H}}_{g,n,P} \rightarrow \overline{\mathcal{M}}_{g,n}$ and its restriction to $p : A_{g,Z,P}^R \rightarrow \overline{\mathcal{M}}_{g,n+m}$. We denote by $\text{Im}(p)$ the image of $A_{g,Z,P}^R$ by p in $\overline{\mathcal{M}}_{g,n+m}$.

Lemma 2.18. *If the vector Z is complete for g and P , then the map $p : A_{g,Z,P}^R \rightarrow \text{Im}(p)$ is a line bundle minus the zero section. In particular $\mathbb{P}A_{g,Z,P}^R$ is isomorphic to its image.*

Proof. Let (C, x_1, \dots, x_{n+m}) be a point of $\text{Im}(p)$. The curve C is smooth and the divisor $\omega_C - \sum_{i=1}^n k_i(x_i) + \sum_{j=1}^m p_j(x_{n+j})$ is a principal divisor of degree 0. Therefore the fiber of p over (C, x_1, \dots, x_{n+m}) is given by the nonzero multiples of one differential with fixed orders of zeros and poles. \square

Definition 2.19. Let $Z = (k_1, \dots, k_n)$ be a vector of nonnegative integer which is not necessarily complete for g and P . A *completion* of Z is a vector $Z' = (k'_1, \dots, k'_{n'})$ such that $n' \geq n$ and for all $1 \leq i \leq n$ we have $k'_i \geq k_i$. We will say that the completion Z' is *exterior* if for all $1 \leq i \leq n$ we have $k'_i = k_i$. Finally we will denote by Z_m the *maximal completion*, i.e. the exterior completion of Z that satisfies $k'_i = 1$ for all $n+1 \leq i \leq n'$.

Lemma 2.20. *We have*

$$A_{g,Z,P}^R = \bigcup_{Z'} \pi(A_{g,Z',P}^R),$$

where the union is over all exterior completions of Z and π is the forgetful map of the zeros that are not accounted for by Z .

Proof. Let $(C, x_1, \dots, x_{n+m}, \alpha)$ be a point of $A_{g,Z,P}^R$. The differential α has zeros exactly of order k_i at the first n marked points, thus the point lies in the image of $A_{g,Z',P}^R$ for an exterior completion. \square

Proposition 2.21. *Let Z be a vector of nonnegative integers and R a vector subspace of \mathcal{R} . The locus $A_{g,Z,P}^R$ is empty or of codimension exactly $|Z| + m - 1 - \dim(R)$ in $\overline{\mathcal{H}}_{g,n,P}$.*

Proof. First we assume that Z is complete. The dimension of $\mathbb{P}A_{g,Z,P}$ is equal to the dimension of its image in the moduli space of curves. We suppose that $R = \mathcal{R}$ (no residue condition). Then the image of $\mathbb{P}A_{g,Z,P}$ is of dimension $2g - 2 + n$ if P is empty (see [25]) and $2g - 3 + n + m$ otherwise (see [12]). By a simple count of dimension we can check that the proposition is valid in this specific case.

We still assume that Z is complete, however we no longer assume that $R = \mathcal{R}$. We have seen that the residue map $A_{g,Z,P}^R \rightarrow R$ is a submersion, therefore the dimension of $A_{g,Z,P}^R$ is equal to the dimension of R plus the dimension of the fiber of the residue map at any point. If we choose $R = \mathcal{R}$, we get that the fiber at any point is given by $\dim A_{g,Z,P} - \dim \mathcal{R}$. Therefore the dimension of $A_{g,Z,P}^R$ is equal to $\dim A_{g,Z,P} - (m - 1) + \dim(R)$. Thus the proposition is valid for all choices of R .

Now, let Z be any vector. Let Z' be an exterior completion. The map $\pi : A_{g,Z',P}^R \rightarrow A_{g,Z,P}^R$ is finite. Indeed the preimage of a point $(C, x_1, \dots, x_{n+m}, \alpha)$ is finite of cardinal $\#\text{Aut}(k'_{n+1}, \dots, k'_{n'})$. The points in the preimage correspond to the different orderings of the zeros that are not accounted for by Z . The proof of Lemma 2.14 implies that if $A_{g,Z',P}$ is not empty for some exterior completion then $A_{g,Z_m,P}$ is not empty: indeed we can always perturbate a differential to “break up” a zero of order greater than 1. By counting the dimensions, we have $\dim(A_{g,Z_m,P}^R) > \dim(A_{g,Z',P}^R)$ for all exterior completions $Z' \neq Z_m$. Therefore $\dim(A_{g,Z_m,P}^R) = \dim(A_{g,Z,P}^R)$ and the proposition is proved. \square

Proposition 2.22. *Let Z be a vector of nonnegative integers. The following statements are equivalent:*

- (1) *there exists a dense open set $U \subset \text{Im}(p)$ such that the fiber of p over any point of U is of dimension 1;*
- (2) *the dimension of $\mathbb{P}A_{g,Z,P}^R$ is less than or equal to the dimension of $\overline{\mathcal{M}}_{g,n+m}$.*

Proof. First, we assume that the dimension of $\mathbb{P}A_{g,Z,P}^R$ is less than or equal to the dimension of $\overline{\mathcal{M}}_{g,n+m}$. Let Z_m be the maximal completion of Z . We have seen

that the image of $A_{g,Z_m,P}^R$ is dense in $A_{g,Z,P}$. Therefore the image of $\text{Im}(p_m)$ is dense in $\text{Im}(p)$:

$$\begin{array}{ccc} A_{g,Z_m,P}^R & \longrightarrow & A_{g,Z_m,P}^R \\ p_m \downarrow & & \downarrow p \\ \text{Im}(p_m) & \longrightarrow & \text{Im}(p). \end{array}$$

In order to prove that the fiber of p over a generic point of $\text{Im}(p)$ is of dimension 1, we only need to prove that $\dim(\text{Im}(p)) = \dim(\mathbb{P}A_{g,Z,P}^R)$. We obviously have $\dim(\mathbb{P}A_{g,Z,P}^R) \geq \dim(\text{Im}(p))$. Now we will prove that $\dim(\mathbb{P}A_{g,Z,P}^R) \leq \dim(\text{Im}(p))$.

We consider the two following vector bundles over the moduli space of curves $\mathcal{M}_{g,n+m}$:

$$\begin{aligned} E_1 &= H^0 \left(C, \omega_C \left(+ \sum_{i=1}^m p_i x_{n+i} \right) \right), \\ E_2 &= \mathcal{R}/R \oplus \left(\bigoplus_{i=1}^n J_{i,k_i} \right), \end{aligned}$$

where J_{i,k_i} is the space of jets of order k_i at the marked point x_i . We have a well defined map $e : E_1 \rightarrow E_2$. The rank of E_1 is $r_1 = g - 1 + \sum p_i$ and the rank of E_2 is $r_2 = m - 1 - \dim(R) + \sum k_i$. By assumption, we have

$$\dim(\mathbb{P}A_{g,Z,P}^R) = \dim(\overline{\mathcal{M}}_{g,n+m}) + r_1 - r_2 - 1 < \dim(\overline{\mathcal{M}}_{g,n+m}).$$

Let $\mathcal{E} \subset \mathcal{M}_{g,n+m}$ be the locus where e is not injective. We have $r_1 < r_2$, thus the locus \mathcal{E} is of codimension at least $r_2 - r_1 + 1$ because it is the vanishing locus of $r_2 - r_1 + 1$ minors of the map e . Therefore the locus \mathcal{E} is of dimension greater than or equal to $\dim(\mathbb{P}A_{g,Z,P}^R)$.

Now we need to prove that $\text{Im}(p)$ is open and dense in \mathcal{E} . Let P' be a vector of m positive integers such that $P' \leq P$. Let Z' be a vector of n nonnegative integers such that $Z' \geq Z$. The image of $\mathbb{P}A_{g,Z',P'}^R$ lies in \mathcal{E} . Conversely, the locus \mathcal{E} is the union of all the $\text{Im}(p')$ where p' is the map from $\mathbb{P}A_{g,Z',P'}^R$ to $\mathcal{M}_{g,n+m}$ for $P' \leq P$ and $Z' \geq Z$. We have $\dim(\mathbb{P}A_{g,Z',P'}^R) < \dim(\mathbb{P}A_{g,Z,P}^R) \leq \dim(\mathcal{E})$ if $P' < P$ or $Z' > Z$. Therefore $\text{Im}(p)$ is open and dense in the locus \mathcal{E} and $\dim(\text{Im}(p)) = \dim(\mathbb{P}A_{g,Z,P}^R)$.

Now to prove the converse implication we assume that the dimension of $\mathbb{P}A_{g,Z,P}^R$ is greater than the dimension of $\overline{\mathcal{M}}_{g,n+m}$. We denote by $d = \dim(\mathbb{P}A_{g,Z,P}^R) - \dim(\overline{\mathcal{M}}_{g,n+m})$. We consider the vector Z_d which is obtained from Z by adding d times 1. We denote $p_d : \mathbb{P}A_{g,Z_d,P}^R \rightarrow \overline{\mathcal{M}}_{g,n+m+d}$ the forgetful map. We have $\dim(\mathbb{P}A_{g,Z_d,P}^R) = \dim(\overline{\mathcal{M}}_{g,n+m+d})$ thus the map p_d is finite on a dense open set of $\overline{\mathcal{M}}_{g,n+m+d}$. Therefore on a dense open subset of $\overline{\mathcal{M}}_{g,n+m}$ the fiber of p contains at least one contracted curve obtained by forgetting the last d marked points. \square

2.6. Stable differentials on disconnected curves. In order to prove the main theorem, we will need a generalization of the notion of stable differentials to the case of disconnected curves. Let q be a positive integer, and

$$\begin{aligned} \mathbf{g} &= (g_1, g_2, \dots, g_q), \\ \mathbf{n} &= (n_1, n_2, \dots, n_q), \\ \mathbf{m} &= (m_1, m_2, \dots, m_q) \end{aligned}$$

be lists of nonnegative integers satisfying $2g_j - 2 + m_j + n_j > 0$. Let

$$\mathbf{P} = (P_j)_{j=1\dots q}$$

be a list of vectors of positive integers of length m_j .

Definition 2.23. The space of stable differentials of type \mathbf{P} is

$$\overline{\mathcal{H}}_{\mathbf{g}, \mathbf{n}, \mathbf{P}} = \prod_{i=1}^q \overline{\mathcal{H}}_{g_i, n_i, P_i}.$$

Proposition 2.24. *The space of stable differentials of type \mathbf{P} is a cone over $\overline{\mathcal{M}}_{\mathbf{g}, \mathbf{n}, \mathbf{m}} \stackrel{\text{def}}{=} \prod_{j=1}^q \overline{\mathcal{M}}_{g_j, n_j + m_j}$. Its Segre class is given by*

$$s(\overline{\mathcal{H}}_{\mathbf{g}, \mathbf{n}, \mathbf{P}}) = \prod_{j=1}^q s(\overline{\mathcal{H}}_{g_j, n_j + m_j, P_j}),$$

where $s(\overline{\mathcal{H}}_{g_j, n_j, P_j})$ is the pull-back of the Segre class of $\overline{\mathcal{H}}_{g_j, n_j, P_j}$ to the product $\prod_{j=1}^q \overline{\mathcal{M}}_{g_j, n_j + m_j}$ under the j^{th} projection.

Proof. The proof is straightforward because the space $\overline{\mathcal{H}}_{\mathbf{g}, \mathbf{n}, \mathbf{P}}$ is a product of cones. \square

To handle the residues, we extend the definition of the space of residues \mathcal{R} :

$$\mathcal{R} = \bigoplus_{j=1}^q \mathcal{R}_j = \{(r_{j,i})_{j,i} \text{ such that } \sum_{i=1}^{m_j} r_{j,i} = 0, \forall j \in [1, q]\} \subset \mathbb{C}^{m_1 + \dots + m_q}.$$

Definition 2.25. Let R be a vector subspace of \mathcal{R} . The space $\overline{\mathcal{H}}_{\mathbf{g}, \mathbf{n}, \mathbf{P}}^R$ is the space of stable differentials with residues lying in R .

Remark 2.26. The linear relations that define the space R can involve residues at poles of different connected components.

Notation 2.27. Let $1 \leq j \leq q$. We will denote by $\text{pr}_j : \mathcal{R} \rightarrow \mathcal{R}_j$ the projection onto \mathcal{R}_j along $\bigoplus_{j' \neq j} \mathcal{R}_{j'}$. We will denote by R_j the space $\text{pr}_j(R)$. The previous remark implies that in general $R \cap \mathcal{R}_j \subsetneq R_j$.

Notation 2.28. Let $\mathbf{Z} = (Z_j)_{j=1\dots q} = ((k_{1,1}, \dots, k_{1,n_1}), \dots, (k_{q,1}, \dots, k_{q,n_q}))$ be a list of vectors of nonnegative integers of length n_j . We define

$$A_{\mathbf{g}, \mathbf{Z}, \mathbf{P}}^R \subset \overline{\mathcal{H}}_{\mathbf{g}, \mathbf{n}, \mathbf{P}}^R$$

as the locus of points $(C, (x_{j,i})_{1 \leq j \leq q, 1 \leq i \leq n_j + m_j}, \alpha) \in \overline{\mathcal{H}}_{\mathbf{g}, \mathbf{n}, \mathbf{P}}^R$ such that C is smooth and α which is nonzero on each connected component and with zeros of orders exactly $k_{j,i}$ at the n_j first marked points of each connected component. If there is no condition on the residues we will simply denote it by $A_{\mathbf{g}, \mathbf{Z}, \mathbf{P}}$. We will call \mathbf{Z} *complete* if all the Z_j are complete.

Lemma 2.29. *Let R be a linear subspace of \mathcal{R} . The space $\overline{\mathcal{H}}_{\mathbf{g}, \mathbf{n}, \mathbf{P}}^R$ is a subcone of $\overline{\mathcal{H}}_{\mathbf{g}, \mathbf{n}, \mathbf{P}}^R$ of codimension $\dim(\mathcal{R}) - \dim(R)$ and we have:*

- the cones $\overline{\mathcal{H}}_{\mathbf{g}, \mathbf{n}, \mathbf{P}}$ and $\overline{\mathcal{H}}_{\mathbf{g}, \mathbf{n}, \mathbf{P}}^R$ have the same Segre class;

- the Poincaré-dual class of $[\mathbb{P}\overline{\mathcal{H}}_{\mathbf{g},\mathbf{n},\mathbf{P}}^R]$ in $H^*(\mathbb{P}\overline{\mathcal{H}}_{\mathbf{g},\mathbf{n},\mathbf{P}}, \mathbb{Q})$ is given by

$$\xi^{\dim(\mathcal{R}) - \dim(R)},$$

- let \mathbf{Z} be a list of vectors which is complete for \mathbf{g} and \mathbf{P} , then the map $p : A_{\mathbf{g},\mathbf{Z},\mathbf{P}}^R \rightarrow R$ is a submersion;
- let \mathbf{Z} be a list of vectors of nonnegative integer. The locus $A_{\mathbf{g},\mathbf{Z},\mathbf{P}}^R$ is empty or of codimension $\sum_{k \in \mathbf{Z}} k + \dim(\mathcal{R}) - \dim(R)$ in $\overline{\mathcal{H}}_{\mathbf{g},\mathbf{n},\mathbf{P}}$.

Proof. All proofs of Section 2.3 and 2.5 can be adapted immediately to the disconnected case. \square

2.7. Fibers of the map $p : A_{\mathbf{g},\mathbf{Z},\mathbf{P}}^R \rightarrow \overline{\mathcal{M}}_{\mathbf{g},\mathbf{Z},\mathbf{P}}$. Let $q \geq 1$. Let $\mathbf{g}, \mathbf{n}, \mathbf{m}$ and \mathbf{P} be lists of integers and vectors of integers of length q as in the previous section. Let \mathcal{R} be the vector of residues. Let \mathbf{Z} be a list of q vectors of nonnegative integers of length prescribed by \mathbf{n} .

If the context is clear, we will denote by the same letter $p : A_{\mathbf{g},\mathbf{Z},\mathbf{P}}^R \rightarrow \overline{\mathcal{M}}_{\mathbf{g},\mathbf{Z},\mathbf{P}}$ the restriction of the map $p : \overline{\mathcal{H}}_{\mathbf{g},\mathbf{n},\mathbf{P}} \rightarrow \overline{\mathcal{M}}_{\mathbf{g},\mathbf{n},\mathbf{P}}$ and by $\text{Im}(p)$ its image. In this section we will state some conditions to determine if the general fiber of p is of dimension 1 or not. This will be important to describe the boundary divisors of the locus $A_{\mathbf{g},\mathbf{Z},\mathbf{P}}^R$.

Let $1 \leq j \leq q$. We denote by p_j the map from $A_{g_j, Z_j, P_j}^{R_j}$ to $\overline{\mathcal{M}}_{g_j, n_j + m_j}$. Finally we denote by $\text{Im}(p_j)$ the image of p_j . We have a natural inclusion of $A_{\mathbf{g},\mathbf{Z},\mathbf{P}}^R$ into $\prod_{j=1}^q A_{g_j, Z_j, P_j}^{R_j}$ and of $\text{Im}(p)$ into $\prod_{i=1}^n \text{Im}(p_j)$.

Assume that \mathbf{Z} is complete. We recall that $A_{g_j, Z_j, P_j}^{R_j} \rightarrow \text{Im}(p_j)$ is a line bundle minus the zero section. We will denote by L_j the pull-back of this line bundle to $\text{Im}(p)$.

We recall that \mathcal{R}_j is the space of residues at the j^{th} component and that we have $\mathcal{R} = \bigoplus_{i=1}^q \mathcal{R}_j$. We define the j -th evaluation map of residues $\text{ev}_j : L_j \rightarrow \mathcal{R}_j$ as the morphism of vector bundles over $\text{Im}(p)$ given by the evaluation of the residues at the i -th connected component. We define the evaluation of residues as the morphism of vector bundles: $\text{ev} = (\bigoplus_{j=1}^q \text{ev}_j) : \bigoplus_{j=1}^q L_j \rightarrow \mathcal{R}$.

Remark 2.30. The evaluation map (ev) and the residue map (res) are not defined on the same spaces. The first one is a morphism of vector bundles on the space $\text{Im}(p)$ while the second one is defined as a morphism of vector bundles over $\mathbb{P}A_{\mathbf{g},\mathbf{Z},\mathbf{P}}^R$. If $q = 1$, then $\mathbb{P}A_{\mathbf{g},\mathbf{Z},\mathbf{P}}^R$ is isomorphic to its image and the two maps correspond.

Proposition 2.31. *Suppose that \mathbf{Z} is complete. Then the families*

$$p : A_{\mathbf{g},\mathbf{Z},\mathbf{P}}^R \rightarrow \text{Im}(p)$$

and

$$p : \text{ev}^{-1}(R) \cap \left(\prod_{j=1}^q L_j^* \right) \rightarrow \text{Im}(p)$$

are isomorphic. If $q \geq 2$, the fiber of p over a point is of dimension 1 if and only if ev is injective and $R \cap \text{ev}(\bigoplus_j L_j)$ is of dimension 1.

Proof. For a point $x \in \text{Im}(p)$, the fiber of p can be described as follows: it is the choice of a nonzero differential for each connected component such that the residues

at the poles define a vector in R . Therefore the fiber over x is the subset of points of $\prod L_j^*$ with residues in R . This fiber is given by $\text{ev}^{-1}(R) \cap \prod_{j=1}^q L_j^*$.

The fiber of $\text{ev}^{-1}(R) \cap \prod_{j=1}^q L_j^*$ over $x \in \text{Im}(p)$ is not empty. Indeed, suppose that for some $1 \leq j \leq q$ the space $\text{ev}^{-1}(R)$ is contained in $\{0\} \oplus_{j' \neq j} L_{j'}$, then the residue condition R imposes that the differential on one of the component is zero. In which case, x is not a point of $\text{Im}(p)$. Therefore the dimension of $\text{ev}^{-1}(R) \cap \prod_{j=1}^q L_j^*$ is the same as the dimension of $\text{ev}^{-1}(R) \cap \bigoplus_{j=1}^q L_j$.

The only point that remains to prove is: if the map ev is not injective then the fiber of p is of dimension greater than 1. We assume that the map ev is not injective, then one of the L_j 's is mapped to zero for some $1 \leq j \leq q$. Thus we have:

$$\text{ev}^{-1}(R) \cap \bigoplus_{j=1}^q L_j = L_j \oplus \left(\text{ev}^{-1}(R) \cap \bigoplus_{j' \neq j} L_{j'} \right).$$

We have seen that $\text{ev}^{-1}(R)$ cannot be contained in $L_j \times \{0\}$, thus the second summands is of positive dimension and $\text{ev}^{-1}(R) \cap \bigoplus_{j=1}^q L_j$ is of dimension greater than 1. \square

Let Σ be the union of the vector subspaces $R \cap \ker(\text{pr}_j)$ for $1 \leq i \leq q$. If R is of positive dimension, we denote by $\mathbb{P}\Sigma$ the image of Σ in $\mathbb{P}R$. This is the locus of vectors of residues that vanish on at least one connected component. Suppose that all R_j are of positive dimension, then $\Sigma \subsetneq R$ and there is a natural map $\rho : \mathbb{P}R \setminus \mathbb{P}\Sigma \rightarrow \prod_{j=1}^q \mathbb{P}R_j$ defined as the projection on each factor.

Notation 2.32. We will say that the residue vector spaces $(\mathcal{R}, R, (\mathcal{R}_j)_{1 \leq i \leq q})$ satisfy the condition (\star) if

- the space R and the R_j 's are of positive dimension;
- the map ρ from $\mathbb{P}R \setminus \mathbb{P}\Sigma$ to $\prod_{i=1}^q \mathbb{P}R_j$ is finite on a dense open subset of $\mathbb{P}R \setminus \mathbb{P}\Sigma$.

Proposition 2.33. *Suppose that \mathbf{Z} is complete and that q is at least 2. Then the fiber of p over a generic point of $\text{Im}(p)$ is of dimension 1 if and only if $(\mathcal{R}, R, (\mathcal{R}_j)_{1 \leq j \leq q})$ satisfy the condition (\star) .*

Proof. We have already seen that if R_j is reduced to the trivial space, then the map $\text{ev} : \bigcup_{j=1}^q L_j \rightarrow \mathcal{R}$ is not injective and the fibers of p are all of dimension greater than 1 (see the proof of Proposition 2.31). We assume that all R_j are non trivial. We denote by $A_j^0 \subset A_{\mathbf{g}, \mathbf{Z}, \mathbf{P}}^R$ to be the locus of differentials with zero residues on the i^{th} component. The image of A_j^0 by the residue map lies in $R \cap \ker(\text{pr}_j)$ which is of positive codimension in R . Besides the residue map is a submersion, thus $\dim(A_j^0) < \dim(A_{\mathbf{g}, \mathbf{Z}, \mathbf{P}}^R)$. We will denote

$$A' = A_{\mathbf{g}, \mathbf{Z}, \mathbf{P}}^R \setminus \bigcup_{j=1}^q A_j^0.$$

The locus A' is dense in $A_{\mathbf{g}, \mathbf{Z}, \mathbf{P}}^R$. If we assume that the fibers of p are generically of dimension 1, then $p(A')$ is also dense in $\text{Im}(p)$. Therefore we only need to prove that a generic point of $p(A')$ has fibers of dimension 1 if and only if condition (\star) is satisfied.

It is easy to check that the residue map sends A' to $R \setminus \Sigma$. Therefore the locus $p(A')$ is the locus of points such that the map ev defined in the proof of Proposition 2.31 is injective. Thus a point of $p(A')$ has fibers of dimension 1 by p if and only if $R \cap \text{ev}(\bigoplus_j L_j)$ is of dimension 1. Now, $R \cap \text{ev}(\bigoplus_j L_j)$ is of dimension 1 if and only if the preimage under ρ of the point $(L_1, \dots, L_q) \in \prod_{j=1}^q \mathbb{P}R_j$ is composed of a unique point.

Now the residue map is a submersion from $A_{\mathbf{g}, \mathbf{Z}, \mathbf{P}}^R$ to R . Therefore, the map ρ is finite on a dense open subset of $\mathbb{P}R \setminus \mathbb{P}\Sigma$ if and only if the fiber of p is of dimension 1 on a dense open set of $\text{Im}(p)$. \square

Now no longer assume that \mathbf{Z} is complete for \mathbf{g} and \mathbf{P} .

Notation 2.34. We will say that $(\mathbf{g}, \mathbf{Z}, \mathbf{P}, R)$ satisfies condition $(\star\star)$ if and only if:

- the vector spaces $(\mathcal{R}, R, (\mathcal{R}_j)_{1 \leq j \leq q})$ satisfy the condition (\star) ;
- for all $1 \leq j \leq q$, we have $\dim(A_{g_j, Z_j, P_j}^{R_j}) - 1 \leq \dim(\overline{\mathcal{M}}_{g_j, n_j + m_j})$.

Proposition 2.35. *Suppose that q is at least 2. The fiber of p over a generic point of $\text{Im}(p)$ is of dimension 1 if and only if $(\mathbf{g}, \mathbf{Z}, \mathbf{P}, R)$ satisfies condition $(\star\star)$.*

Proof. Let \mathbf{Z}_m be the maximal completion of \mathbf{Z} . We denote by $Z_{j,m}$ the maximal completion of Z_j . We recall that

$$\dim(A_{\mathbf{g}, \mathbf{Z}_m, \mathbf{P}}^R) = \dim(A_{\mathbf{g}, \mathbf{Z}, \mathbf{P}}^R).$$

If $(R, \mathcal{R}, (\mathcal{R}_j)_{1 \leq j \leq q})$ does not satisfy the condition (\star) , then the dimension of $A_{\mathbf{g}, \mathbf{Z}_m, \mathbf{P}}^R$ is greater than the dimension of its image in the moduli space of curves and the general fiber of p is of dimension greater than 1. From now on, we assume that $(R, \mathcal{R}, (\mathcal{R}_j)_{1 \leq j \leq q})$ satisfies the condition (\star) .

First, we suppose that $\dim(A_{g_j, Z_j, P_j}^{R_j}) - 1 > \dim(\overline{\mathcal{M}}_{g_j, n_j + m_j})$ for some $j \in [1, q]$. The preimage of a point in $\text{Im}(p_j)$ under p_j has fibers of dimension greater than 1. Let y be a point in $A_{\mathbf{g}, \mathbf{Z}, \mathbf{P}}^R$. The point y determines point $y_{j'}$ in $A_{g_{j'}, Z_{j'}, P_{j'}}^{R_{j'}}$ for $j' \neq j$ and a point y_j in $A_{g_j, Z_j, P_j}^{R_j}$. We denote by E_j the locus of points in $y_j' \in p_j^{-1}(\{p_j(y_j)\})$ such that the residues of y_j' and y_j are equal. By hypothesis, the locus E_j is of dimension at least 1. Now the preimage of $p(y)$ under p contains the points $(y_1, \dots, y_j', \dots, y_q)$ for all $y_j' \in E_j$. Therefore the dimension of the fiber of $p(y)$ under p is greater than 1.

Now, we assume that $\dim(A_{g_j, Z_j, P_j}^{R_j}) - 1 \leq \dim(\overline{\mathcal{M}}_{g_j, n_j + m_j})$ for all $1 \leq j \leq q$. We already know that $\dim(\text{Im}(p)) \leq \dim(A_{\mathbf{g}, \mathbf{Z}, \mathbf{P}}^R)$. To prove that $\dim(\text{Im}(p)) \geq \dim(A_{\mathbf{g}, \mathbf{Z}, \mathbf{P}}^R)$, we study the two vector bundles over $\overline{\mathcal{M}}_{\mathbf{g}, \mathbf{n}, \mathbf{m}}$ defined as:

$$\begin{aligned} E_1 &= \bigoplus_{j=1}^q H^0 \left(C_j, \omega_{C_j} \left(+ \sum_{i=1}^{m_j} p_i x_{n_j+i} \right) \right), \\ E_2 &= \mathcal{R}/R \oplus \left(\bigoplus_{j=1}^q \bigoplus_{i=1}^{n_q} J_{j,i,k_i} \right), \end{aligned}$$

where J_{j,i,k_i} is the space of jets of order k_i at the i^{th} marked point of the j^{th} component. There is a natural morphism of vector bundles $e : E_1 \rightarrow E_2$. As in the

proof of Proposition 2.22, $\text{Im}(p)$ is dense in the locus where e is not injective and we conclude that $\dim(A_{g_j, Z_j, P_j}^{R_j}) - 1 = \dim(\overline{\mathcal{M}}_{g_j, n_j + m_j})$. \square

2.8. Unstable base. Here we extend the definition of the spaces of stable differentials to the case of an unstable base. Two kinds of spaces $\overline{\mathcal{H}}_{g, n, P}$ with unstable base appear as degenerations of the spaces with stable base:

- The genus g is equal to 0, $n = m = 1$ and $P = (p)$ with $p > 1$ (stability condition).
- The genus g is equal to 0, $n = 0$, $m = 2$ and $P = (1, p)$ with $p > 1$ (stability condition).

These two spaces are not cones because they have no geometric bases; however we can still define these moduli spaces and the \mathbb{C}^* -action. We begin by parametrizing the second space. Given $\alpha \in \overline{\mathcal{H}}_{0, 0+2, (1, p)}$, there exists a global coordinate w on the rational curve such that

$$\alpha = [w^{p-1} + a_1 w^{p-2} + \dots + a_{p-1}] \frac{dw}{w}.$$

The global coordinate w is unique up to the multiplication by a $(p-1)$ -st root of unity ζ . Thus the space $\overline{\mathcal{H}}_{0, 0+2, (1, p)}$ is equal to $\mathbb{C}^{p-1}/\mathbb{Z}_{p-1}$ where the coefficients (a_1, \dots, a_{p-1}) are the coordinates of \mathbb{C}^{p-1} and the action of \mathbb{Z}_{p-1} is given by $\zeta \cdot (a_1, \dots, a_{p-1}) = (\zeta a_1, \dots, \zeta^{p-1} a_{p-1})$.

Let $\lambda \in \mathbb{C}^*$ and let $\eta \in \mathbb{C}^*$ such that $\eta^{p-1} = \lambda$. The action of \mathbb{C}^* on $\overline{\mathcal{H}}_{0, 0+2, (1, p)}$ is given by the multiplication $\alpha \rightarrow \lambda \alpha$. We have

$$\lambda \alpha = [(\eta w)^{p-1} + a_1 \eta (\eta w)^{p-2} + \dots + a_{p-1}] \frac{d(\eta w)}{\eta w}.$$

Therefore \mathbb{C}^* acts on $\mathbb{C}^{p-1}/\mathbb{Z}_{p-1}$ with weights $\frac{j}{p-1}$ on the coordinate a_j . Thus we can define the projectivization of $\overline{\mathcal{H}}_{0, 0+2, (1, p)}$, which comes with a tautological line bundle in the orbifold sense.

The space $\overline{\mathcal{H}}_{0, 1+1, (p)}$ is constructed in the same way by taking $a_{p-1} = 0$. It is the subspace of $\overline{\mathcal{H}}_{0, 0+2, (1, p)}$ composed of differentials with zero residues. It is also the complementary of the space of principal parts in the space of generalized principal parts (see Section 2.1).

Definition 2.36. A triple (g, n, P) composed of a nonnegative integers g and n and a vector P of positive integers is *semi-stable* if $2g - 2 + n + \ell(P) > 0$ or $g = 0$, $n = 1$ and $P = (p)$ with $p > 1$ or $g = 0$, $n = 0$, $P = (1, p)$ with $p > 1$.

If Z is vector of nonnegative integers, then the triple (g, Z, P) is *semi-stable* if $(g, \ell(Z), P)$ is semi-stable.

Consider two lists \mathbf{g} and \mathbf{n} of q nonnegative integers and a list \mathbf{P} of q vectors. The triple $(\mathbf{g}, \mathbf{n}, \mathbf{P})$ is *semi-stable* if all (g_j, n_j, P_j) are semi-stable. If \mathbf{Z} is a list of vectors then $(\mathbf{g}, \mathbf{Z}, \mathbf{P})$ is *semi-stable* if all (g_j, Z_j, P_j) are semi-stable.

Definition 2.37. Let $\mathbf{g}, \mathbf{n}, \mathbf{m}, \mathbf{P}$ be lists of genera, numbers of marked points without poles, numbers of marked poles and vectors of positive integers indexed by $j \in \llbracket 1, q \rrbracket$. We suppose that $(\mathbf{g}, \mathbf{n}, \mathbf{P})$ is a semi-stable triplet. We define $\overline{\mathcal{H}}_{\mathbf{g}, \mathbf{n}, \mathbf{P}} = \prod_{j=1}^q \overline{\mathcal{H}}_{g_j, n_j, P_j}$. If at least one of the $j \in \llbracket 1, q \rrbracket$ satisfies $2g_j - 2 + n_j + m_j > 0$, then $\overline{\mathcal{H}}_{\mathbf{g}, \mathbf{n}, \mathbf{P}}$ is a cone over the following base

$$\overline{\mathcal{M}}_{\mathbf{g}, \mathbf{n}, \mathbf{m}}^{\text{red}} = \prod_{j: 2g_j - 2 + n_j + m_j > 0} \overline{\mathcal{M}}_{g_j, n_j + m_j}.$$

We will call this space, the *reduced base*.

Now we can extend the definition of the previous sections to semi-stable triples.

Notation 2.38. Let \mathbf{Z} be a list of vectors such that the triple $(\mathbf{g}, \mathbf{Z}, \mathbf{P})$ is semi-simple. Let $R \subset \mathcal{R}$ a vector subspace. We still denote by $A_{\mathbf{g}, \mathbf{Z}, \mathbf{P}}^R \subset \overline{\mathcal{H}}_{\mathbf{g}, \mathbf{n}, \mathbf{P}}$ the locus of differentials with residues lying in R and zeros of order prescribed by \mathbf{Z} (see Notation 2.28).

We also define the tautological ring.

Definition 2.39. Let $p : \overline{\mathcal{H}}_{\mathbf{g}, \mathbf{n}, \mathbf{P}} \rightarrow \overline{\mathcal{M}}_{\mathbf{g}, \mathbf{n}, \mathbf{m}}^{\text{red}}$ be the projection to the base. The *tautological ring* $RH^*(\mathbb{P}\overline{\mathcal{H}}_{\mathbf{g}, \mathbf{n}, \mathbf{P}})$ is the ring generated by $\xi = c_1(\%_0(1))$ and pull-backs by p of tautological classes from the base $\overline{\mathcal{M}}_{\mathbf{g}, \mathbf{n}, \mathbf{m}}^{\text{red}}$.

We generalize Theorem 1.

Theorem 4. For all $\mathbf{g}, \mathbf{Z}, \mathbf{P}$ (list of integers and vectors of integers) and R a subspace of \mathcal{R} , the Poincaré-dual class of $\mathbb{P}A_{\mathbf{g}, \mathbf{Z}, \mathbf{P}}^R$ in $H^*(\mathbb{P}\overline{\mathcal{H}}_{\mathbf{g}, \mathbf{n}, \mathbf{P}}, \mathbb{Q})$ is tautological and can be explicitly computed.

2.9. Pre-stable graphs. Let $\mathbf{g}, \mathbf{n}, \mathbf{m}, \mathbf{P}$ be lists of genera, numbers of marked points without poles, numbers of marked poles and vectors of positive integers indexed by $j \in \llbracket 1, q \rrbracket$ as in the previous Section. The space $\overline{\mathcal{H}}_{\mathbf{g}, \mathbf{n}, \mathbf{P}}$ has a natural stratification whose strata are described by pre-stable graphs that we define here.

Definition 2.40. A *pre-stable graph* of type $(\mathbf{g}, \mathbf{n}, \mathbf{m}, \mathbf{P})$ is given by the data

$$\Gamma = (V, H, g : V \rightarrow \mathbb{N}, a : H \rightarrow V, i : H \rightarrow H, E),$$

satisfying the following properties:

- V is a vertex set with a genus function g .
- H is a half-edge set equipped with a vertex assignment a and an involution i ;
- the edge set E is defined as the set of length 2 orbits of i in H (self-edges at vertices are permitted);
- (V, E) has q connected components;
- for all $1 \leq j \leq q$, the genus of the connected component labeled by j is defined by $\sum g(v) + \#(E_j) - \#(V_j) + 1$ and is equal to g_j ;
- L is the set of fixed points of i called *legs*;
- for all $1 \leq j \leq q$, there are $n_j + m_j$ legs on the j^{th} connected component;
- for each vertex v in V ,
 - let $n(v)$ be the number of legs of v corresponding to marked points without poles;
 - let $m(v)$ be the number of legs of v corresponding to marked points with poles plus the number of half-edges adjacent to v which correspond to edges;
 - let $P(v)$ be the vector of orders of poles at marked points adjacent to v , to which we add poles of order one for the half-edges;
- for each vertex v , the triple $(g(v), n(v), P(v))$ is semi-stable.

We define the following lists indexed by the vertices of Γ :

$$\begin{aligned} \mathbf{g}_\Gamma &= (g(v))_{v \in V} & , & \quad \mathbf{n}_\Gamma = (n(v))_{v \in V}, \\ \mathbf{m}_\Gamma &= (m(v))_{v \in V} & , & \quad \mathbf{P}_\Gamma = (P(v))_{v \in V}. \end{aligned}$$

The triple $(\mathbf{g}_\Gamma, \mathbf{n}_\Gamma, \mathbf{P}_\Gamma)$ by definition of a pre-stable graph. We will consider the space $\overline{\mathcal{H}}_{\mathbf{g}_\Gamma, \mathbf{n}_\Gamma, \mathbf{P}_\Gamma}$. We denote by \mathcal{R}_Γ the space of residues of $\overline{\mathcal{H}}_{\mathbf{g}_\Gamma, \mathbf{n}_\Gamma, \mathbf{P}_\Gamma}$. We define $R_\Gamma \subset \mathcal{R}$ as the vector subspace of residues satisfying that the sum of residues at two half edges of an edge is zero.

Notation 2.41. Let Γ be a pre-stable graph we denote by $\overline{\mathcal{H}}_\Gamma$ the moduli space $\overline{\mathcal{H}}_{\mathbf{g}_\Gamma, \mathbf{n}_\Gamma, \mathbf{P}_\Gamma}^{R_\Gamma}$ and by

$$\zeta_\Gamma^\# : \overline{\mathcal{H}}_\Gamma \rightarrow \overline{\mathcal{H}}_{\mathbf{g}, \mathbf{n}, \mathbf{P}}$$

the natural closed immersion.

Thus boundary strata of $\overline{\mathcal{H}}_{\mathbf{g}, \mathbf{n}, \mathbf{P}}$ are described by pre-stable graphs.

The space $\mathbb{P}\overline{\mathcal{H}}_\Gamma$ comes with a tautological line bundle. This line bundle is the pullback by $\zeta_\Gamma^\#$ of the tautological line bundle of $\mathbb{P}\overline{\mathcal{H}}_{\mathbf{g}, \mathbf{n}, \mathbf{P}}$. By abuse of notation we will write ξ for the first Chern class of the dual of the tautological line bundle for both spaces. We have the following important proposition.

Proposition 2.42. *Let Γ be pre-stable graph. The morphism $\zeta_{\Gamma*}^\# : H^*(\mathbb{P}\overline{\mathcal{H}}_\Gamma, \mathbb{Q}) \rightarrow H^*(\mathbb{P}\overline{\mathcal{H}}_{\mathbf{g}, \mathbf{n}, \mathbf{P}}, \mathbb{Q})$ maps tautological classes to tautological classes.*

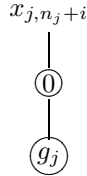
Proof. Let Γ be a pre-stable graph. Let $k \geq 0$ and $\beta \in \overline{\mathcal{M}}_\Gamma^{\text{red}}$. We need to prove that the class $\zeta_{\Gamma*}^\#(\xi^k p^*(\beta))$ is tautological. We will prove this statement in three steps.

Stable graphs. We suppose first that Γ is a stable graph. We recall that in this case we have defined a map $\zeta_\Gamma : \overline{\mathcal{M}}_\Gamma \rightarrow \overline{\mathcal{M}}_{\mathbf{g}, \mathbf{n}, \mathbf{m}}$. Then $\overline{\mathcal{H}}_\Gamma$ is the fiber product

$$\begin{array}{ccc} \overline{\mathcal{H}}_\Gamma & \xrightarrow{\zeta_\Gamma^\#} & \overline{\mathcal{H}}_{\mathbf{g}, \mathbf{n}, \mathbf{P}} \\ p_\Gamma \downarrow & & \downarrow p \\ \overline{\mathcal{M}}_\Gamma & \xrightarrow{\zeta_\Gamma} & \overline{\mathcal{M}}_{\mathbf{g}, \mathbf{n}, \mathbf{m}} \end{array}$$

Let β be a cohomology class in $H^*(\overline{\mathcal{M}}_\Gamma, \mathbb{Q})$. We use the projection formula and the fact that $\overline{\mathcal{H}}_\Gamma$ is a fiber product to get $\zeta_{\Gamma*}^\#(\xi^k \cdot p_\Gamma^*(\beta)) = \xi^k p^*(\zeta_{\Gamma*}(\beta))$. Therefore, if the class β belongs to the tautological ring $RH^*(\overline{\mathcal{M}}_\Gamma, \mathbb{Q})$, then the class $\zeta_{\Gamma*}^\#(\xi^k \cdot p_\Gamma^*(\beta))$ belongs to the tautological ring of $\overline{\mathcal{H}}_{\mathbf{g}, \mathbf{n}, \mathbf{P}}$.

Graph with one main vertex. Now we no longer assume that Γ is stable. Let $1 \leq j \leq q$ and $1 \leq i \leq m_j$. Let p_i be the i^{th} entry of P_j . Assume that Γ is the following graph



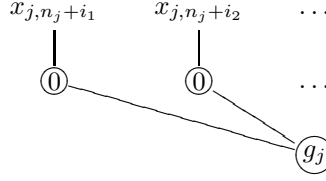
(we take the trivial graph for all the other connected components). We will prove that the class $\zeta_{\Gamma*}^{\#}(1)$ lies in $RH^*(\mathbb{P}\overline{\mathcal{H}}_{\mathbf{g},\mathbf{n},\mathbf{m},\mathbf{P}})$. We use the parametrization of the cone of principal parts at x

$$\left[\left(\frac{u}{z} \right)^{p_i-1} + a_1 \left(\frac{u}{z} \right)^{p_i-2} + \dots + a_{p_i-2} \left(\frac{u}{z} \right) \right] \frac{dz}{z}.$$

The stratum defined by Γ is the vanishing locus of u . We have seen that u^{p_i-1} is a section of the line bundle $\text{Hom}(\%_0(-1), \mathcal{L}_i^{p_i-1})$. Therefore the vanishing locus of u has Poincaré-dual class given by

$$[u=0] = \frac{1}{p_i-1} \xi - \psi_i.$$

By the same argument, if Γ is the graph



where the set $\{i_k\}$ is a set of indices in $\llbracket 1, m_j \rrbracket$. Then we have

$$\zeta_{\Gamma*}^{\#}(1) = \prod_k \left(\frac{1}{p_{i_k}-1} \xi - \psi_{i_k} \right).$$

And more generally, for a class β in $RH^*(\overline{\mathcal{M}}_{\mathbf{g},\mathbf{n},\mathbf{m},\mathbf{P}}^{\text{red}})$ and $k \in \mathbb{N}$, we have

$$\zeta_{\Gamma*}^{\#}(\xi^k \beta) = \xi^k \beta \cdot \prod_k \left(\frac{1}{p_{i_k}-1} \xi - \psi_{i_k} \right) \in RH^*(\overline{\mathcal{H}}_{\mathbf{g},\mathbf{n},\mathbf{m},\mathbf{P}}).$$

General unstable graph. We combine the two previous arguments. Let Γ be a general pre-stable graph. Let $\hat{\Gamma}$ be the graph obtained by contracting all edges between stable vertices. We have $\overline{\mathcal{M}}_{\mathbf{g},\mathbf{n},\mathbf{m}}^{\text{red}} = \overline{\mathcal{M}}_{\hat{\Gamma}}^{\text{red}}$. The space $\overline{\mathcal{H}}_{\Gamma}$ is the fiber product

$$\begin{array}{ccccc} \overline{\mathcal{H}}_{\Gamma} & \longrightarrow & \overline{\mathcal{H}}_{\hat{\Gamma}}^{\text{red}} & \xrightarrow{\zeta_{\hat{\Gamma}}^{\#}} & \overline{\mathcal{H}}_{\mathbf{g},\mathbf{n},\mathbf{m}} \\ \text{\scriptsize p_{Γ}} \downarrow & & \downarrow \text{\scriptsize $p_{\hat{\Gamma}}$} & & \\ \overline{\mathcal{M}}_{\Gamma} & \xrightarrow{\zeta_{\Gamma}} & \overline{\mathcal{M}}_{\mathbf{g},\mathbf{n},\mathbf{m}}^{\text{red}} & & \end{array}$$

Thus $\zeta_{\Gamma*}^{\#}(\xi^k p_{\Gamma}^* \beta) = \zeta_{\hat{\Gamma}*}^{\#}(\xi^k p_{\hat{\Gamma}}^*(\zeta_{\Gamma*} \beta))$. Now $\hat{\Gamma}$ has one stable vertex, and $\zeta_{\hat{\Gamma}*} \beta \in RH^*(\overline{\mathcal{M}}_{\mathbf{g},\mathbf{n},\mathbf{m}}^{\text{red}})$ thus the class $\zeta_{\Gamma*}^{\#}(\xi^k p_{\Gamma}^* \beta)$ is tautological. \square

3. THE INDUCTION FORMULA

The aim of this section is to prove Theorem 4 stating that the cohomology classes $[\mathbb{P}\overline{A}_{\mathbf{g},\mathbf{Z},\mathbf{P}}^R] \in H^*(\mathbb{P}\overline{\mathcal{H}}_{\mathbf{g},\mathbf{n},\mathbf{P}}, \mathbb{Q})$ are tautological. For this purpose we will state and prove the most technical and important result: the induction formula for the classes $[\mathbb{P}\overline{A}_{\mathbf{g},\mathbf{Z},\mathbf{P}}^R]$. Before doing this, we need to understand the closure of $A_{\mathbf{g},\mathbf{Z},\mathbf{P}}^R$ in $\overline{\mathcal{H}}_{\mathbf{g},\mathbf{n},\mathbf{P}}$. We will describe this closure using graphs with twists and level structures following [2] and [12]. However our space $\overline{\mathcal{H}}_{\mathbf{g},\mathbf{n},\mathbf{P}}$ is different from the spaces used in [2] and [12] so that we have to modify their definitions. That is why we will introduce **P**-admissible graphs.

3.1. Twisted graphs with level structures. Let $\mathbf{g}, \mathbf{n}, \mathbf{m}, \mathbf{P}$ be lists of genera, numbers of marked points without poles, numbers of marked poles and vectors of positive integers indexed by $j \in \llbracket 1, q \rrbracket$. We suppose that the triplet $(\mathbf{g}, \mathbf{n}, \mathbf{P})$ is semi-stable. Let \mathbf{Z} be a list of q vectors of nonnegative integers of lengths prescribed by \mathbf{n} . Let R be a vector subspace of the space of residues \mathcal{R} . We introduce a stratification of $\overline{A}_{\mathbf{g},\mathbf{Z},\mathbf{P}}^R$. Strata will be described by combinatorial objects called **P**-admissible graphs. We introduce these graphs here and explain how they correspond to strata of $\overline{A}_{\mathbf{g},\mathbf{Z},\mathbf{P}}^R$ in section 3.2.

Let Γ be a pre-stable graph of type $(\mathbf{g}, \mathbf{n}, \mathbf{m}, \mathbf{P})$. We denote by H_e the set of half-edges of Γ which are not legs.

Definition 3.1. A *twist* on Γ is a function

$$I : H_e \rightarrow \mathbb{Z}$$

Satisfying the following conditions.

- If h and h' form an edge, then $I(h) + I(h') = 0$.
- Let v and v' be two vertices, and $\{(h_1, h'_1), \dots, (h_n, h'_n)\}$ be the set of edges from v to v' . Then either $I(h_j) = 0$ for all $1 \leq j \leq n$, or $I(h_j) > 0$ for all $1 \leq j \leq n$, or $I(h_j) < 0$ for all $1 \leq j \leq n$. We say that $v = v'$, or $v > v'$, or $v < v'$, depending on the above inequalities.
- The relation \leq thus defined on vertices is transitive.

For shortness, a pre-stable graph endowed with a twist function will be called a *twisted graph*. If (Γ, I) is a twisted graph, the above conditions define a partial order on the set of its vertices of Γ .

Definition 3.2. A *level structure* on a twisted graph is a function:

$$l : \text{Vertices} \rightarrow \mathbb{Z}^-,$$

compatible with the partial order induced by the twist, i.e., for all vertices v and v' ,

$$v = v' \Rightarrow l(v) = l(v'), \quad v < v' \Rightarrow l(v) < l(v').$$

We impose that the image of l is an interval containing all integers from 0 to $-d$ and we call d the *depth* of the twisted graph. We will denote by V^i the set of vertices of level i .

Definition 3.3. A twisted graph with level structure is called **P**-*admissible* if all marked poles of order at least 2 belong to vertices of level 0. For shortness we will call such graphs *admissible graphs*.

Example 3.4. We represent in Figure 1 an example of admissible graphs. Each vertex v is represented by a circle containing the integer g_v . The marked poles and zeros are represented by legs. A leg corresponding to a pole (respectively a zero) of order k is marked by $-k$ (respectively $+k$). The twists are indicated on each edge.

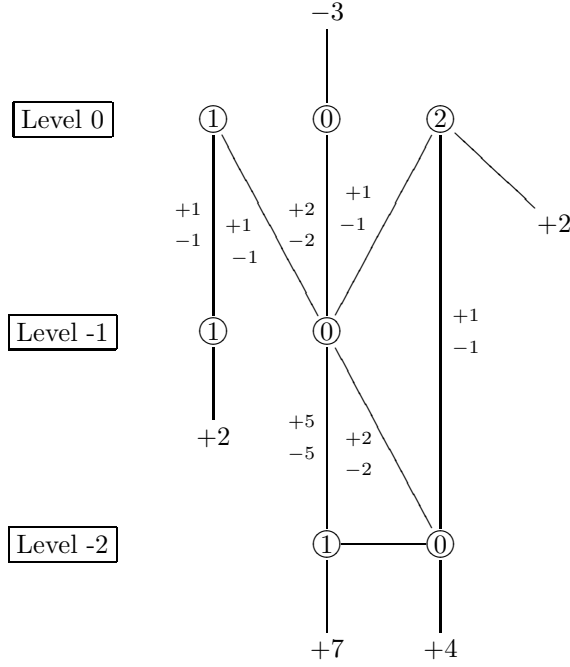


FIGURE 1. An example of admissible graph for the vectors $Z = (2, 2, 4, 7)$ and $P = (-3)$.

Definition 3.5. An edge between vertices of the same level will be called an *horizontal edge*.

3.2. Boundary strata associated to admissible graphs. Let $\mathbf{g}, \mathbf{n}, \mathbf{m}, \mathbf{P}, \mathbf{Z}$ and R be as in the previous section. Let (Γ, I, l) be an admissible graph. In this subsection, we assign to this admissible graph a stratum of abelian differentials $A_{\Gamma, I, l} \subset \overline{\mathcal{H}}_{\mathbf{g}, \mathbf{n}, \mathbf{P}}$ that lies in the closure of $A_{\mathbf{g}, \mathbf{Z}, \mathbf{P}}^R$. We build this stratum level by level.

Level 0. Suppose there are q_0 level 0 vertices. Their genera, half-edges and twists determine lists $\mathbf{g}_0, \mathbf{n}_0, \mathbf{m}_0, \mathbf{P}_0, \mathbf{Z}_0$ of length q_0 as in Section 2.6: half-edges h to deeper levels are listed as zeros of orders $I(h) - 1$.

Now we define a space of residues. Residues are assigned to legs that correspond to marked poles and to horizontal edges. These residues should satisfy three conditions:

- the residues on the two half-edges of a horizontal edge are opposite;
- the sum of residues at each vertex vanishes;
- the vector of residues on the marked poles belongs to the space R .

These conditions define a vector space denoted by R^0 . With these data, we define the level 0 stratum $A_{\Gamma,I,l}^0 = A_{\mathbf{g}_0, \mathbf{P}_0, \mathbf{Z}_0}^{R^0}$.

Level -1. Suppose there are q_1 level -1 vertices. Their genera, half-edges and twists determine lists $\mathbf{g}_1, \mathbf{n}_1, \mathbf{m}_1, \mathbf{P}_1, \mathbf{Z}_1$ of length q_1 as in Section 2.6: half-edges to level 0 are listed as poles of order $-I(h) + 1$ and half-edges to deeper levels as zeros of order $I(h) - 1$.

To define the space of residue R^1 we need a little more notation than for the level 0. We define a first space of residues $\tilde{\mathcal{R}}$: residues are assigned to legs that correspond to marked poles on components of level 0 or -1 and to edges between components of level 0 or -1. The space $\tilde{\mathcal{R}}$ is the direct sum of \mathcal{R}^0 and \mathcal{R}^1 corresponding respectively to residues assigned to half-edges of level 0 and -1. Let pr be the projection from $\tilde{\mathcal{R}}$ to \mathcal{R}^1 along \mathcal{R}^0 . Let $\tilde{R}^1 \subset \tilde{\mathcal{R}}$ be the vector subspace defined by the linear conditions:

- the residues on the two half-edges of a horizontal edge are opposite;
- the sum of residues at each vertex vanishes;
- the vector of residues on the marked poles on components of level 0 belongs to the space R .

The vector space R^1 is defined as $\text{pr}(\tilde{R}^1)$. The vector space R^1 can be characterized by the following straightforward lemma:

Lemma 3.6. *Let $\phi : \mathbb{C} \rightarrow \overline{\mathcal{H}}_\Gamma$ be a holomorphic family of differentials such that: ϕ vanishes identically on components of level < -1 and $\phi(0)$ vanishes on components of level -1 then the image of $\phi'(0)$ by the tangent map of the evaluation of residues at level -1 lies in R^1 .*

Now we define

$$A_{\Gamma,I,l}^1 = p(A_{\mathbf{g}_1, \mathbf{Z}_1, \mathbf{P}_1}^{R^1}) \subset \overline{\mathcal{M}}_{\mathbf{g}_1, \mathbf{n}_1, \mathbf{m}_1} = \prod_{v \in V^1} \overline{\mathcal{M}}_{g_v, n_v + m_v},$$

where $p : \overline{\mathcal{H}}_{\mathbf{g}_1, \mathbf{n}_1, \mathbf{P}_1} \rightarrow \overline{\mathcal{M}}_{\mathbf{g}_1, \mathbf{n}_1, \mathbf{m}_1}$ is the forgetful map.

Level $-\ell$. Let Γ' be the graph obtained from Γ by contracting edges between vertices of levels 0 through $-\ell + 1$. The twist on Γ restricts to Γ' and the level structure is shifted. Vertices of levels 0 to $-\ell + 1$ merge to level 0, level $-\ell$ vertices become level -1 vertices and so on. Now $A_{\Gamma,I,l}^\ell$ is defined as $A_{\Gamma',I,l}^1$.

Notation 3.7. Now that we have defined the $A_{\Gamma,I,l}^\ell$ for all levels, we denote

$$A_{\Gamma,I,l} = \prod_{\ell \in \mathbb{Z}^-} A_{\Gamma,I,l}^\ell.$$

We have a natural inclusion of $A_{\Gamma,I,l} \hookrightarrow \overline{\mathcal{H}}_{\mathbf{g}, \mathbf{n}, \mathbf{P}}$: the differential is nonzero only on the level 0 vertices and vanishes identically everywhere else. The following lemma states that the $A_{\Gamma,I,l}$ give a stratification of the locus $\overline{\mathcal{A}}_{\mathbf{g}, \mathbf{Z}, \mathbf{P}}^R$.

Lemma 3.8. *Let (Γ, I, l) be an admissible graph. The locus $A_{\Gamma,I,l}$ lies in the closure of $A_{\mathbf{g}, \mathbf{Z}, \mathbf{P}}^R$. Conversely if y is a point of $\overline{\mathcal{A}}_{\mathbf{g}, \mathbf{Z}, \mathbf{P}}^R$ then there exists a completion \mathbf{Z}' of \mathbf{Z} and an admissible graph (Γ, I, l) such that y lies in $\pi(A_{\Gamma,I,l})$, where $\pi : A_{\mathbf{g}, \mathbf{Z}', \mathbf{P}}^R \rightarrow A_{\mathbf{g}, \mathbf{Z}, \mathbf{P}}^R$ is the forgetful map of the marked zeros that are accounted for by \mathbf{Z} .*

For admissible graphs of depth 1, the proof of “ $A_{\Gamma,I,l}$ lies in the closure of $A_{\mathbf{g},\mathbf{Z},\mathbf{P}}^R$ ” will be part of the proof of Theorem 5 in Section 3.5. We will only prove here that we can deduce the first part of Lemma 3.8 for all admissible graphs from the case of admissible graphs of depth 1.

Proof. We prove the first statement of the lemma by induction on the depth of the admissible graph. Let (Γ, I, l) be an admissible graph of depth $d > 1$. Let (Γ', I', l') be the admissible graph of depth $d - 1$ obtained by merging the level 0 and -1 together. We consider the locus $A_{\Gamma', I', l'}^0$ defined above. Let (Γ^0, I^0, l^0) be the graph obtained by keeping only the vertices of levels 0 and -1 . The graph (Γ^0, I^0, l^0) is of depth 1. Applying the lemma to the graph (Γ^0, I^0, l^0) we get that A_{Γ^0, I^0, l^0} lies in the closure of $A_{\Gamma', I', l'}^0$. Therefore $A_{\Gamma, I, l}$ lies in the closure of $A_{\Gamma', I', l'}$. By induction hypothesis, the stratum $A_{\Gamma', I', l'}$ lies in the closure of $A_{\mathbf{g}, \mathbf{Z}, \mathbf{P}}^R$, thus $A_{\Gamma, I, l}$ lies in the closure of $A_{\mathbf{g}, \mathbf{Z}, \mathbf{P}}^R$. \square

The proof of the converse statement is made in [2]: see Theorem 1.3 and the proof of “the conditions are necessary” from page 33 to page 35. We only need to say a few words about the difference of compactification. In [2], the authors consider the moduli space

$$K\overline{\mathcal{M}}_{g,n,P} = \left\{ (C, x_1, \dots, x_{n+m}, \alpha) : \begin{array}{l} (C, x_1, \dots, x_{n+m}) \in \overline{\mathcal{M}}_{g,n+m}, \\ \alpha \in H^0(C, \omega_C(-\sum_{i=1}^m p_i x_{n+i})) \end{array} \right\},$$

which is a vector bundle over $\overline{\mathcal{M}}_{g,n+m}$. The incidence variety compactification is equal to the closure of the strata

$$\Omega\mathcal{M}_g^{\text{inc}}(Z - P) = \left\{ \begin{array}{l} (C, x_1, \dots, x_{n+m}, \alpha) : C \text{ smooth,} \\ \alpha \in H^0(C, \omega_C(\sum_{i=1}^n k_i x_i - \sum_{i=1}^m p_i x_{n+i})) \end{array} \right\}$$

in $K\overline{\mathcal{M}}_{g,n,P}$. If Z is complete then the two compactifications (incidence variety and our compactification by stable differentials) are topologically isomorphic. The isomorphism between the two spaces is defined by

$$\begin{aligned} F : \overline{A}_{g,Z,P} &\rightarrow \Omega\mathcal{M}_g^{\text{inc}}(Z - P) \\ (C, x_1, \dots, x_{n+m}, \alpha) &\mapsto (\text{stab}(C), x_1, \dots, x_{n+m}, \tilde{\alpha}), \end{aligned}$$

where $\text{stab}(C)$ is the stabilization of C and $\tilde{\alpha}$ is the differential obtained by restriction after contraction of the unstable components.

Indeed in the compactification by stable differentials, poles are not allowed to belong to components of level < 0 . However, if Z is complete then an unstable component has necessarily a pole of order k and a zero of order $k - 2$ at a node with a deeper component. The space $A_{0,(k-2),(k)}$ is a single point. Thus, by contracting all unstable components we get a point in $\mathbb{P}\Omega\overline{\mathcal{M}}^{\text{inc}}(Z - P)$ (see Example 5.3).

Now if \mathbf{Z} is not complete then we deduce the lemma from the fact that

$$A_{\mathbf{g},\mathbf{Z},\mathbf{P}}^R = \bigcup_{\mathbf{Z}'} \pi(A_{\mathbf{g},\mathbf{Z}',\mathbf{P}}^R)$$

where the sum goes over all exterior completions of \mathbf{Z} .

Remark 3.9. If Z is complete then the two compactifications differ only by the orbifold structure. However if Z is not complete then the two topologies no longer correspond.

3.3. Description of boundary divisors. Let $\mathbf{g}, \mathbf{Z}, \mathbf{P}$ and R be as in the previous sections. In the proof of the main theorem, we will be interested in the vanishing loci of sections of certain line bundles over $\overline{A}_{\mathbf{g}, \mathbf{Z}, \mathbf{P}}^R$. That is why we need to understand the boundary divisors of $\overline{A}_{\mathbf{g}, \mathbf{Z}, \mathbf{P}}^R$. The purpose of this section is to determine the set of admissible graphs which are associated to strata of codimension 1, that is to divisors.

Lemma 3.10. *Let (Γ, I, l) be an admissible graph. The codimension of $\overline{A}_{\Gamma, I, l}$ in $\overline{A}_{\mathbf{g}, \mathbf{Z}, \mathbf{P}}^R$ is greater than or equal to the depth d of the level structure l .*

Proof. As we have seen in the proof of Lemma 3.8, the stratum described by a graph with level structure lies in the closure of the stratum obtained by merging levels 0 and -1 of the graph. Thus every time we merge two levels we decrease the codimension at least by 1. \square

Lemma 3.11. *Let (Γ, I, l) be an admissible graph of depth 1. The codimension of $\overline{A}_{\Gamma, I, l}$ in $\overline{A}_{\mathbf{g}, \mathbf{Z}, \mathbf{P}}^R$ is greater than the number of horizontal edges.*

Proof. We can independently merge vertices along horizontal edges. At every merging, we decrease the codimension by at least 1. \square

It follows from Lemmas 3.10 and 3.11 that a nontrivial admissible graph corresponding to a divisor is necessarily of depth 1 and has no horizontal edges.

Notation 3.12. We denote by $\text{Bic}(\mathbf{g}, \mathbf{Z}, \mathbf{P}, R)$ the set of graphs with two levels and possessing no horizontal edges. We will call such graphs *bicolor graphs*.

Remark 3.13. Elements of $\text{Bic}(\mathbf{g}, \mathbf{Z}, \mathbf{P}, R)$ are twisted graphs with level structures. However, the level structure of a bicolored graph is completely determined by the twists. This is why we will denote now (Γ, I) the elements of $\text{Bic}(\mathbf{g}, \mathbf{Z}, \mathbf{P}, R)$.

We recall from Section 3.2 that the boundary strata associated to a graph of depth 1 is equal to $p(A_{\mathbf{g}_1, \mathbf{Z}_1, \mathbf{P}_1}^{R^1}) \times A_{\mathbf{g}_0, \mathbf{Z}_0, \mathbf{P}_0}^{R^0}$, where p is the map from $A_{\mathbf{g}_1, \mathbf{Z}_1, \mathbf{P}_1}^{R^1}$ to the moduli space of curves $\mathcal{M}_{\mathbf{g}_1, \mathbf{n}_1, \mathbf{m}_1}$.

Proposition 3.14. *Let (Γ, I) be a bicolor graph. We assume that $\overline{A}_{\Gamma, I, l}$ is nonempty. Then $\overline{A}_{\Gamma, I, l}$ is a divisor of $\overline{A}_{\mathbf{g}, \mathbf{Z}, \mathbf{P}}^R$ if and only if $(\mathbf{g}_1, \mathbf{Z}_1, \mathbf{P}_1, R^1)$ satisfies the condition $(\star\star)$.*

Proof. We have the equality

$$\dim(\overline{A}_{\mathbf{g}, \mathbf{Z}, \mathbf{P}}^R) = \dim(\overline{A}_{\Gamma, I, l}) + \dim(A_{\mathbf{g}_1, \mathbf{Z}_1, \mathbf{P}_1}^{R^1}) - \dim(p(A_{\mathbf{g}_1, \mathbf{Z}_1, \mathbf{P}_1}^{R^1})).$$

Therefore the stratum $\overline{A}_{\mathbf{g}, \mathbf{Z}, \mathbf{P}}^R$ is a divisor if and only if the fibers of p are of dimension 1. Thus the proposition is a direct consequence of Proposition 2.35. \square

Notation 3.15. We denote by $\text{Div}(\mathbf{g}, \mathbf{Z}, \mathbf{P}, R)$ the set of bicolor graphs (Γ, I) such that $\overline{A}_{\Gamma, I, l}$ is a divisor of $\overline{A}_{\mathbf{g}, \mathbf{Z}, \mathbf{P}}^R$. For short we will call these elements *admissible divisor graphs*.

Notation 3.16. Let $1 \leq j \leq q$ and $1 \leq i \leq \ell(Z_j)$. We denote by $\mathbf{Z}_{\ell, i}$ the list of vector obtained from \mathbf{Z} by increasing the i^{th} coordinate of Z_ℓ by one.

Proposition 3.17. *Let \mathbf{Z}' be a completion of \mathbf{Z} and (Γ, I, l) be an admissible graph such that $D = \pi(\overline{A}_{\Gamma, I})$ is a divisor of $\overline{A}_{\mathbf{g}, \mathbf{Z}, \mathbf{P}}^R$ (where π is the forgetful map of the points), then D is necessarily of one of the three kinds:*

- (1) *the stratum $\overline{A}_{\Gamma, I}$ for $(\Gamma, I) \in \text{Div}(\mathbf{g}, \mathbf{Z}, \mathbf{P}, R)$;*
- (2) *the locus $\overline{A}_{\mathbf{g}, \mathbf{Z}_{\ell, i}, \mathbf{P}}^R$ for some label (j, i) corresponding to a marked point which is not a pole;*
- (3) *the locus $\overline{A}_{\Gamma, I, l}$ for a \mathbf{P} -admissible graph of depth 0 with a unique horizontal edge.*

Proof. Let \mathbf{Z}' be a completion of \mathbf{Z} . If \mathbf{Z}' is not the maximal completion then $\dim(A_{\mathbf{g}, \mathbf{Z}', \mathbf{P}}^R) < \dim(A_{\mathbf{g}, \mathbf{Z}, \mathbf{P}}^R)$. The only possible graph is the trivial one and this corresponds to type 2. Now if $\mathbf{Z}' = \mathbf{Z}_m$, then (Γ, I, l) is of depth less than 1: if (Γ, I, l) is of depth 1 then it is a divisor graph (type 1) and if (Γ, I, l) has depth 0 then (Γ, I, l) has at most one horizontal edge (type 3). \square

Proposition 3.18. *Let D_1 and D_2 be two divisors obtained from an admissible graph as in Proposition 3.17. Then D_1 and D_2 have no common irreducible components.*

Proof. The divisors D_1 and D_2 can be of one of the three types described in Proposition 3.17. We will prove this proposition by considering every possible cases.

Type 1/type 1. Let (Γ, I) and (Γ', I') in $\text{Div}(\mathbf{g}, \mathbf{Z}, \mathbf{P}, R)$ such that $A_{\Gamma, I}$ and $A_{\Gamma', I'}$ have a common irreducible component D . The component D determines a pre-stable graph by taking the dual graph of a any point of $D \cap A_{\Gamma, I}$, therefore $\Gamma = \Gamma'$. Moreover, the vertices of Γ with identically zero differentials are the vertices of level -1 . Therefore the level structure (or more precisely the signs of the twists) are the same for (Γ, I) and (Γ', I') . Now the twist at an edge is determined by the vanishing order of the differential at the corresponding node on the component of level 0 for any point in $D \cap A_{\Gamma, I}$. Therefore $(\Gamma, I) = (\Gamma', I')$. Thus divisors of type 1 have no common irreducible components.

Type 2. The underlying generic curve of the divisors of type 2 is a curve without singularities, therefore divisors of type 2 do not intersect divisors of type 1 or type 3. Now the differentials of the generic differentials of two divisors of type 2 have different vanishing order at two of the marked points

Type 3. Two divisors of type 3 are distinguished by the topological types of a generic curve. Besides, a divisor of type 3 is distinguished from a divisor of type 1 because none of the components carries a vanishing differential in a divisor of type 3. \square

Locus of generic points. Let $(\Gamma, I) \in \text{Div}(\mathbf{g}, \mathbf{Z}, \mathbf{P}, R)$. We recall that

$$A_{\Gamma, I} = p(A_{\mathbf{g}_1, \mathbf{Z}_1, \mathbf{P}_1}^{R^1}) \times A_{\mathbf{g}_0, \mathbf{Z}_0, \mathbf{P}_0}^{R^0},$$

where $p : A_{\mathbf{g}_1, \mathbf{Z}_1, \mathbf{P}_1}^{R^1} \rightarrow \overline{\mathcal{M}}_{\mathbf{g}_1, \mathbf{n}_1, \mathbf{m}_1}$ is the forgetful map. The condition $(\star\star)$ ensures that there exists an open dense locus $A_1^{\text{gen}} \subset A_{\mathbf{g}_1, \mathbf{Z}_1, \mathbf{P}_1}^{R^1}$ such that the map $p : A_1^{\text{gen}} \rightarrow p(A_1^{\text{gen}})$ has fibers of dimension 1 (see Proposition 2.35). Then we set

$$A_{\Gamma, I}^{\text{gen}} = A_1^{\text{gen}} \times A_{\mathbf{g}_0, \mathbf{Z}_0, \mathbf{P}_0}^{R^0}.$$

This open locus of generic points will be important for us because the map

$$p : A_1^{\text{gen}} \times A_{\mathbf{g}_0, \mathbf{Z}_0, \mathbf{P}_0}^{R^0} \rightarrow A_{\Gamma, I}^{\text{gen}} = p(A_1^{\text{gen}}) \times A_{\mathbf{g}_0, \mathbf{Z}_0, \mathbf{P}_0}^{R^0}$$

is a line bundle minus the zero section.

Notation 3.19. We denote by $p : \mathcal{N}_{\Gamma, I} \rightarrow A_{\Gamma, I}^{\text{gen}}$ this line bundle.

3.4. Class and multiplicity of a boundary divisor. Let $\mathbf{g}, \mathbf{Z}, \mathbf{P}$ and R be as in the previous sections. We want to compute the Poincaré-dual class of the divisor associated to an element of $\text{Div}(\mathbf{g}, \mathbf{P}, \mathbf{Z}, R)$.

Let (Γ, I) be an admissible graph in $\text{Bic}(\mathbf{g}, \mathbf{P}, \mathbf{Z}, R)$ (this graph is a divisor or not). We recall that

$$A_{\Gamma, I} \simeq p(A_{\mathbf{g}_1, \mathbf{Z}_1, \mathbf{P}_1}^{R^1}) \times A_{\mathbf{g}_0, \mathbf{Z}_0, \mathbf{P}_0}^{R^0} \subset \overline{\mathcal{M}}_{\mathbf{g}_1, \mathbf{n}_1, \mathbf{m}_1} \times \overline{\mathcal{H}}_{\mathbf{g}_0, \mathbf{n}_0, \mathbf{P}_0}.$$

Now, we recall that the prestable graph Γ determines a stratum

$$\zeta_{\Gamma}^{\#} : \overline{\mathcal{H}}_{\Gamma} = \overline{\mathcal{H}}_{\mathbf{g}_{\Gamma}, \mathbf{n}_{\Gamma}, \mathbf{P}_{\Gamma}}^{R_{\Gamma}} \rightarrow \overline{\mathcal{H}}_{\mathbf{g}, \mathbf{n}, \mathbf{P}}$$

(see Section 2.9). We define the linear subspace $\overline{R}_{\Gamma} \subset R_{\Gamma}$, as the space of vectors of residues such that the residues at poles on components of level -1 (including the edges) are equal to zero. Moreover, we denote by

$$d_{\Gamma} = \dim(R_{\Gamma}) - \dim(\overline{R}_{\Gamma}).$$

The Poincaré-dual class of $\mathbb{P}\overline{\mathcal{H}}_{\mathbf{g}_{\Gamma}, \mathbf{n}_{\Gamma}, \mathbf{P}_{\Gamma}}^{R_{\Gamma}}$ in $H^*(\mathbb{P}\overline{\mathcal{H}}_{\Gamma}, \mathbb{Q})$ is equal to $\xi^{d_{\Gamma}}$ (see Lemma 2.10). Moreover we have

$$\overline{\mathcal{H}}_{\mathbf{g}_{\Gamma}, \mathbf{n}_{\Gamma}, \mathbf{m}_{\Gamma}, \mathbf{P}_{\Gamma}}^{R_{\Gamma}} \simeq \overline{\mathcal{H}}_{\mathbf{g}_0, \mathbf{n}_0, \mathbf{P}_0} \times \left(\prod_{v \in V^1} \overline{\mathcal{H}}_{g_v, n_v + m_v} \right),$$

where the poles at vertices of level 0 are the marked poles of $\overline{\mathcal{H}}_{\mathbf{g}, \mathbf{n}, \mathbf{P}}$ restricted to each vertex and the half-edges are counted as marked points without poles; the spaces at the vertices of level -1 are the spaces of *holomorphic* differentials. We consider the projection

$$\begin{array}{ccc} \mathcal{O}(-1) & \longrightarrow & \prod_{v \in V_1} p_v^*(\overline{\mathcal{H}}_{g_v, n_v + m_v}) \\ \downarrow & \swarrow & \\ \mathbb{P}\overline{\mathcal{H}}_{\mathbf{g}_{\Gamma}, \mathbf{n}_{\Gamma}, \mathbf{m}_{\Gamma}, \mathbf{P}_{\Gamma}}^{R_{\Gamma}} & & \end{array}$$

where p_v is the forgetful map from $\overline{\mathcal{H}}_{g_v, n_v + m_v}$ to $\overline{\mathcal{M}}_{g_v, n_v + m_v}$. Therefore the Poincaré-dual class of the locus of differentials with vanishing differential on the level -1 in $H^*(\mathbb{P}\overline{\mathcal{H}}_{\Gamma}, \mathbb{Q})$ is given by

$$\xi^{d_{\Gamma}} \cdot \prod_{v \in V_1} (\xi^{g_v} + \lambda_1 \xi^{g_v - 1} + \dots + \lambda_{g_v}).$$

We denote this locus by $\mathbb{P}\tilde{A}_{\Gamma, I} \subset \mathbb{P}\overline{\mathcal{H}}_{\Gamma}$. We have a natural identification:

$$\mathbb{P}\tilde{A}_{\Gamma, I} \simeq \mathbb{P}\overline{\mathcal{H}}_{\mathbf{g}_0, \mathbf{n}_0, \mathbf{P}_0} \times \overline{\mathcal{M}}_{\mathbf{g}_1, \mathbf{n}_1, \mathbf{P}_1}.$$

We denote by Φ_0 and Φ_1 the projections on both factors.

Definition 3.20. The class $a_{\Gamma, I} \in H^*(\mathbb{P}\overline{\mathcal{H}}_{\mathbf{g}, \mathbf{n}, \mathbf{P}}, \mathbb{Q})$ is given by

$$\zeta_{\Gamma}^{\#} \left(\xi^{d_{\Gamma}} \cdot \Phi_1^*(p_*[\mathbb{P}\overline{A}_{\mathbf{g}_1, \mathbf{Z}_1, \mathbf{P}_1}^{R^1}]) \cdot \Phi_0^*[\mathbb{P}\overline{A}_{\mathbf{g}_0, \mathbf{Z}_0, \mathbf{P}_0}^{R^0}] \prod_{v \in V^1} (\xi^{g_v} + \lambda_1 \xi^{g_v - 1} + \dots + \lambda_{g_v}) \right).$$

Proposition 3.21. Let $(\Gamma, I) \in \text{Bic}(\mathbf{g}, \mathbf{P}, \mathbf{Z}, R)$. We have:

- (1) if (Γ, I) is divisor graph then $a_{\Gamma, I} = [\mathbb{P}\bar{A}_{\Gamma, I}]$;
- (2) If (Γ, I) is not a divisor graph then $a_{\Gamma, I} = 0$;
- (3) if $[\mathbb{P}\bar{A}_{\mathbf{g}_0, \mathbf{Z}_0, \mathbf{P}_0}^{R^0}]$ and $[\mathbb{P}\bar{A}_{\mathbf{g}_1, \mathbf{Z}_1, \mathbf{P}_1}^{R^1}]$ are tautological and can be explicitly computed then so is $a_{\Gamma, I}$.

Proof of the first and second points. If (Γ, I) is a divisor graph then $p : A_{\mathbf{g}_1, \mathbf{Z}_1, \mathbf{P}_1}^{R^1} \rightarrow \text{Im}(p)$ is of degree 1, thus $p_*[A_{\mathbf{g}_1, \mathbf{Z}_1, \mathbf{P}_1}^{R^1}] = [p(A_{\mathbf{g}_1, \mathbf{Z}_1, \mathbf{P}_1}^{R^1})]$. Therefore, by construction $a_{\Gamma, I}$ is the Poincaré-dual class of $\mathbb{P}\bar{A}_{\Gamma, I}$.

If (Γ, I) belongs to $\text{Bic}(\mathbf{g}, \mathbf{P}, \mathbf{Z}, R) \setminus \text{Div}(\mathbf{g}, \mathbf{P}, \mathbf{Z}, R)$ then the fibers of $p : \mathbb{P}A_{\mathbf{g}_1, \mathbf{Z}_1, \mathbf{P}_1}^{R^1} \rightarrow \text{Im}(p)$ are of positive dimension and $p_*[\mathbb{P}\bar{A}_{\mathbf{g}_1, \mathbf{Z}_1, \mathbf{P}_1}^{R^1}] = 0$. \square

Proof of the third point. We assume that $[\mathbb{P}\bar{A}_{\mathbf{g}_0, \mathbf{Z}_0, \mathbf{P}_0}^{R^0}]$ and $[\mathbb{P}\bar{A}_{\mathbf{g}_1, \mathbf{Z}_1, \mathbf{P}_1}^{R^1}]$ are tautological and can be explicitly computed.

The projections Φ_1 is equal to the composition of the forgetful map from $\bar{\mathcal{H}}_\Gamma$ to $\bar{\mathcal{M}}_\Gamma^{\text{red}}$ with the projection to the vertices of level -1 . Thus by definition, if β is a tautological class of $\bar{\mathcal{M}}_{\mathbf{g}_1, \mathbf{Z}_1, \mathbf{m}_1}$ then $\Phi_1^*\beta$ is a tautological class of $H^*(\mathbb{P}\bar{\mathcal{H}}_{\mathbf{g}, \mathbf{Z}, \mathbf{P}}, \mathbb{Q})$. Besides, if $[\mathbb{P}\bar{A}_{\mathbf{g}_1, \mathbf{Z}_1, \mathbf{P}_1}^{R^1}]$ is tautological and be explicitly computed then so is $p_*[\mathbb{P}\bar{A}_{\mathbf{g}_1, \mathbf{Z}_1, \mathbf{P}_1}^{R^1}]$: indeed the Segre class of $\bar{\mathcal{H}}_{\mathbf{g}_1, \mathbf{Z}_1, \mathbf{P}_1}$ is a tautological class of $\bar{\mathcal{M}}_{\mathbf{g}_1, \mathbf{m}_1, \mathbf{P}_1}$.

The map Φ_1 is equivariant with respect to the \mathbb{C}^* -action, thus we have $\Phi_1^{-1}(c_1(\%_0(1))) = c_1(\%_0(1))$. Besides the following diagram commutes:

$$\begin{array}{ccc} \mathbb{P}\tilde{A}_{\Gamma, I} & \xrightarrow{\Phi_0} & \mathbb{P}\bar{A}_{\mathbf{g}_0, \mathbf{n}_0, \mathbf{P}_0} \\ \downarrow & & \downarrow p \\ \bar{\mathcal{M}}_\Gamma^{\text{red}} & \longrightarrow & \bar{\mathcal{M}}_{\mathbf{g}_0, \mathbf{n}_0, \mathbf{m}_0}^{\text{red}} \end{array}$$

Thus, if β is a tautological class of $\bar{\mathcal{M}}_{\mathbf{g}_0, \mathbf{n}_0, \mathbf{m}_0}^{\text{red}}$, then the class $\Phi_0^*(p^*(\beta))$ is a tautological class of $\mathbb{P}\bar{\mathcal{H}}_\Gamma$ and thus a tautological class of $H^*(\mathbb{P}\bar{\mathcal{H}}_{\mathbf{g}, \mathbf{n}, \mathbf{P}}, \mathbb{Q})$. \square

Definition 3.22. Let $(\Gamma, I) \in \text{Bic}(\mathbf{g}, \mathbf{P}, \mathbf{Z}, R)$. The *multiplicity* of (Γ, I)

$$m(I) = \prod_{h \rightarrow V^0} I(h),$$

where the product runs over the half-edges which are not legs, pointing to vertices of level 0. The *least commune multiple* and the *group of roots* of the twist are

$$\begin{aligned} L(I) &= \text{LCM}(\{I(h)\}_{h \rightarrow V^0}), \\ G_I &= \left(\prod_{h \rightarrow V^0} \mathbb{Z}_{I(h)} \right) / \mathbb{Z}_{L(I)}. \end{aligned}$$

Let $1 \leq j \leq q$ and $1 \leq i \leq n_j$. Let $k_{i,j}$ be the i^{th} entry of Z_j . We consider the line bundle:

$$\mathcal{O}(-1) \otimes \mathcal{L}_{j,i}^{k_{j,i}+1} \Big|_{A_{\mathbf{g}, \mathbf{P}, \mathbf{Z}}^R} \simeq \text{Hom} \left(\mathcal{O}(-1), \mathcal{L}_{j,i}^{k_{j,i}+1} \right) \Big|_{A_{\mathbf{g}, \mathbf{P}, \mathbf{Z}}^R},$$

where $\mathcal{L}_{j,i}$ is the cotangent line bundle to the i -th marked point of j -th connected component. Let $s_{j,i}$ be the natural section of the line bundle $\text{Hom}(\mathcal{O}(-1), \mathcal{L}_{j,i}^{k_{j,i}+1})|_{A_{\mathbf{g}, \mathbf{P}, \mathbf{Z}}^R}$.

which maps a differential to its $(k_{j,i} + 1)^{\text{st}}$ -order term at the i^{th} marked point of the j^{th} connected component.

Lemma 3.23. *Then the section $s_{j,i}$ vanishes with multiplicity 1 along $\mathbb{P}\overline{A}_{\mathbf{g}, \mathbf{Z}_{j,i}, \mathbf{P}}^R$.*

Proof. Let y_0 be a point of $\mathbb{P}A_{\mathbf{g}, \mathbf{Z}_{j,i}, \mathbf{P}}^R$. We have seen in the proof of Lemma 2.14 that a neighborhood U of y_0 in $\mathbb{P}A_{\mathbf{g}, \mathbf{Z}_{j,i}, \mathbf{P}}^R$ is parametrized by $H^1(\Sigma \setminus \{x_{n+1}, \dots, x_{n+m}\}, \{x_1, \dots, x_n\}; \mathbb{C})$ and that a neighborhood of y_0 in $\mathbb{P}\overline{\mathcal{H}}_{\mathbf{n}, \mathbf{m}, \mathbf{P}}$ is parametrized by $H \times \prod_{i=1}^n \mathcal{Z}_{j,i}$ where $\mathcal{Z}_{j,i}$ is an open neighborhood of 0 in $\mathbb{C}^{k_{j,i}-1}$.

Thus a neighborhood of U in $\mathbb{P}A_{\mathbf{g}, \mathbf{Z}, \mathbf{P}}^R$ is parametrized by $U \times \Delta$ where Δ is an open disk of \mathbb{C} with parameter ϵ . Indeed a deformation of an element of $\mathbb{P}A_{\mathbf{g}, \mathbf{Z}_{j,i}, \mathbf{P}}^R$ will determine a coordinate z and a complex number ϵ such that the differential is given by $d(z^{k_{j,i}+1}(z + \epsilon))$ in a neighborhood of $x_{j,i}$. This choice is unique up to multiplication of z by a $(k_{j,i} + 2)^{\text{nd}}$ -root of unity. Fix a choice for the coordinate z then we have $s_{j,i}(u, \epsilon) = \epsilon$. Thus the vanishing multiplicity of $s_{j,i}$ along $\mathbb{P}A_{\mathbf{g}, \mathbf{Z}_{j,i}, \mathbf{P}}^R$ is equal to 1. \square

Notation 3.24. We denote by $\text{Bic}(\mathbf{g}, \mathbf{P}, \mathbf{Z}, R)_{j,i} \subset \text{Bic}(\mathbf{g}, \mathbf{P}, \mathbf{Z}, R)$ the subset of bicolor graphs such that the i^{th} marked point of the j^{th} connected component belongs to a level -1 vertex and we will denote by

$$\text{Div}(\mathbf{g}, \mathbf{P}, \mathbf{Z}, R)_{j,i} = \text{Bic}(\mathbf{g}, \mathbf{P}, \mathbf{Z}, R)_{j,i} \cap \text{Div}(\mathbf{g}, \mathbf{P}, \mathbf{Z}, R).$$

Lemma 3.25. *The divisors contained in the vanishing locus of $s_{j,i}$ are exactly the divisors corresponding to admissible graphs in $\text{Div}(\mathbf{g}, \mathbf{P}, \mathbf{Z}, R)_{j,i}$ and the divisor $\mathbb{P}\overline{A}_{\mathbf{g}, \mathbf{Z}_{j,i}, \mathbf{P}}^R$. No two of these divisors have a common irreducible component.*

Proof. It is a consequence of Propositions 3.17 and 3.18. \square

The first Chern class of $\mathcal{O}(-1) \otimes \mathcal{L}_{j,i}^{k_{j,i}+1}$ is equal to $\xi + (k_{j,i} + 1)\psi_{j,i}$ therefore we deduce from Lemmas 3.23 and 3.25 that

$$(\xi + (k_{j,i} + 1)\psi_{j,i}) \cdot [\mathbb{P}\overline{A}_{\mathbf{g}, \mathbf{Z}, \mathbf{P}}^R] = [\mathbb{P}\overline{A}_{\mathbf{g}, \mathbf{Z}_{j,i}, \mathbf{P}}^R] + \mathbb{Z},$$

where \mathbb{Z} is a cycle supported on the union of $\mathbb{P}\overline{A}_{\Gamma, I}$ for $(\Gamma, I) \in \text{Div}(\mathbf{g}, \mathbf{P}, \mathbf{Z}, R)_{j,i}$. In the next section, we will prove that $\mathbb{Z} = \sum_{(\Gamma, I) \in \text{Div}(\mathbf{g}, \mathbf{P}, \mathbf{Z}, R)_{j,i}} m(I) a_{\Gamma, I}$ where the sums runs over all (Γ, I) in $\text{Div}(\mathbf{g}, \mathbf{P}, \mathbf{Z}, R)_{j,i}$. We can already remark that Proposition 3.21 implies the following

Corollary 3.26. *The following equality holds:*

$$\sum_{(\Gamma, I) \in \text{Bic}(\mathbf{g}, \mathbf{P}, \mathbf{Z}, R)_{j,i}} m(I) a_{\Gamma, I} = \sum_{(\Gamma, I) \in \text{Div}(\mathbf{g}, \mathbf{P}, \mathbf{Z}, R)_{j,i}} m(I) a_{\Gamma, I}.$$

Proof. It follows from the fact that if $(\Gamma, I) \in \text{Bic}(\mathbf{g}, \mathbf{P}, \mathbf{Z}, R)_{j,i} \setminus \text{Div}(\mathbf{g}, \mathbf{P}, \mathbf{Z}, R)_{j,i}$ then $a_{\Gamma, I} = 0$. \square

3.5. Induction formula. With the above notation, we state the main theorem of the article.

Let $\mathbf{g} = (g_1, \dots, g_q)$, $\mathbf{Z} = (Z_1, \dots, Z_q)$, $\mathbf{P} = (P_1, \dots, P_q)$ be lists of nonnegative integers (genera), vectors of nonnegative integers (orders of zeros), and vectors of positive integers (orders of poles), respectively. Let \mathbf{n} and \mathbf{m} be two lists of q nonnegative integers given by $n_j = \text{length}(Z_j)$, $m_j = \text{length}(P_j)$. Let $R \subset \mathcal{R}$ be

a space of residue conditions. We assume that $(\mathbf{g}, \mathbf{Z}, \mathbf{P})$ satisfy the semi-stability condition of Definition 2.36.

Let $1 \leq j \leq q$ and let $1 \leq i \leq n_j$. Let $k_{j,i}$ be the i th element of Z_j . Denote by $\mathbf{Z}_{j,i}$ the list of vectors obtained from \mathbf{Z} by increasing k_i by 1. Denote by $\psi_{j,i} \in H^2(\mathbb{P}\overline{\mathcal{H}}_{\mathbf{g},\mathbf{n},\mathbf{P}}, \mathbb{Q})$ the ψ -class corresponding to the i th marked point on the j th connected component of the curve.

Recall that to each bicolored graph $(\Gamma, I) \in \text{Bic}(\mathbf{g}, \mathbf{P}, \mathbf{Z}, R)_{j,i}$ we have assigned a cohomology class $\alpha_{\Gamma,I} \in H^*(\mathbb{P}\overline{\mathcal{H}}_{\mathbf{g},\mathbf{n},\mathbf{P}}, \mathbb{Q})$ and a positive integer $m(I)$ (see Section 3.4).

Theorem 5. *In $H^*(\mathbb{P}\overline{\mathcal{H}}_{\mathbf{g},\mathbf{n},\mathbf{P}}, \mathbb{Q})$ we have*

$$(1) \quad [\mathbb{P}\overline{A}_{\mathbf{g},\mathbf{Z}_{j,i},\mathbf{P}}^R] = (\xi + (k_{j,i} + 1)\psi_{j,i}) \cdot [\mathbb{P}\overline{A}_{\mathbf{g},\mathbf{Z},\mathbf{P}}^R] - \sum_{(\Gamma,I) \in \text{Bic}(\mathbf{g},\mathbf{P},\mathbf{Z},R)_{j,i}} m(I) \alpha_{\Gamma,I}$$

if $2g_j - 2 + n_j + m_j > 0$, or

$$(2) \quad [\mathbb{P}\overline{A}_{\mathbf{g},\mathbf{Z}_{j,1},\mathbf{P}}^R] = \frac{p-k-2}{p-1} \xi \cdot [\mathbb{P}\overline{A}_{\mathbf{g},\mathbf{Z},\mathbf{P}}^R]$$

if $g_j = 0$, $Z_j = (k)$, $P_j = (p)$.

Proof of (1). First of all, by Corollary 3.26 we replace the sum over bicolored graphs in Equation (1) by a sum over divisor graphs, i.e. elements of $\text{Div}(\mathbf{g}, \mathbf{Z}, \mathbf{P}, R)_{j,i}$. We will prove the equality in this form.

As in Section 3.4 consider the line bundle $\text{Hom}(\mathcal{O}(-1), \mathcal{L}_{j,i}^{k_{j,i}+1}) \rightarrow \mathbb{P}\overline{A}_{\mathbf{g},\mathbf{Z},\mathbf{P}}^R$. Its first Chern class equals $\xi + (k_{j,i} + 1)\psi_{j,i}$. Moreover, it has a natural section $s_{j,i}$ which maps a differential to its $(k_{j,i} + 1)$ st-order term at the marked point (j, i) .

In Lemma 3.8 we announced that the locus $\mathbb{P}A_{\Gamma,I}$ lies in the closure of $\mathbb{P}\overline{A}_{\mathbf{g},\mathbf{Z},\mathbf{P}}^R$, however we postponed the proof. In Lemma 3.23 we showed that $s_{j,i}$ vanishes along $\mathbb{P}\overline{A}_{\mathbf{g},\mathbf{Z}_{j,i},\mathbf{P}}^R$ with multiplicity 1. In Lemma 3.25 we showed that the remaining vanishing loci of $s_{j,i}$ are supported on the $\mathbb{P}\overline{A}_{\Gamma,I}$ for (Γ, I) of $\text{Div}(\mathbf{g}, \mathbf{Z}, \mathbf{P}, R)_{j,i}$. Now we claim that the vanishing order of $s_{j,i}$ along the locus $\mathbb{P}A_{\Gamma,I}$ is equal to $m(I)$ (see Definition 3.22). Both the closure statement with postponed proof and the multiplicity statement are contained in Lemma 3.27 below. Thus Equation (1) will follow immediately from this lemma. \square

We recall that for a divisor graph $(\Gamma, I) \in \text{Div}(\mathbf{g}, \mathbf{Z}, \mathbf{P}, R)_{j,i}$ we have (see Section 3.2)

$$A_{\Gamma,I} = p(A_{\mathbf{g}_1,\mathbf{Z}_1,\mathbf{P}_1}^{R^1}) \times A_{\mathbf{g}_0,\mathbf{Z}_0,\mathbf{P}_0}^{R^0},$$

where

$$p : A_{\mathbf{g}_1,\mathbf{Z}_1,\mathbf{P}_1}^{R^1} \rightarrow \overline{\mathcal{M}}_{\mathbf{g}_1,\mathbf{n}_1,\mathbf{m}_1}$$

is the forgetful map (see Section 3.3). Moreover we have defined an open dense subset of generic points $A_1^{\text{gen}} \subset A_{\mathbf{g}_1,\mathbf{Z}_1,\mathbf{P}_1}^{R^1}$ such that $p(A_1^{\text{gen}})$ is dense and open in $p(A_{\mathbf{g}_1,\mathbf{Z}_1,\mathbf{P}_1}^{R^1})$ and the map $p : A_1^{\text{gen}} \rightarrow p(A_1^{\text{gen}})$ is a line bundle minus the zero section. We denote by

$$p : \mathcal{N}_{\Gamma,I} \rightarrow A_{\Gamma,I}^{\text{gen}}$$

the pull-back of this line bundle to $A_{\Gamma,I}^{\text{gen}} = p(A_1^{\text{gen}}) \times A_{\mathbf{g}_0,\mathbf{Z}_0,\mathbf{P}_0}^{R^0}$.

Also recall the group G_I and the least common multiple $L(I)$ assigned to the set of twists in Section 3.4.

Lemma 3.27. *Let (Γ, I) be a divisor graph in $\text{Div}(\mathbf{g}, \mathbf{Z}, \mathbf{P}, R)_{j,i}$. Let $y_0 \in \mathbb{P}A_{\Gamma, I}^{\text{gen}}$. Let Δ be an open disk in \mathbb{C} containing 0 and parametrized by ϵ . There exists an open neighborhood U of y_0 in $\mathbb{P}A_{\Gamma, I}^{\text{gen}}$ together with a map $\iota : U \times \Delta \times G_I \rightarrow \mathbb{P}\overline{\mathcal{H}}_{\mathbf{g}, \mathbf{n}, \mathbf{m}, \mathbf{P}}$ satisfying:*

- *the restriction $\iota|_{U \times 0 \times g}$ is the identity on U for all $g \in G_I$;*
- *the image of the restriction $\iota|_{\epsilon \neq 0}$ lies in the open stratum $\mathbb{P}A_{\mathbf{g}, \mathbf{Z}, \mathbf{P}}^R$;*
- *for all $g \in G_I$, the section $s_{j,i}$ restricted to $\iota(U \times \Delta \times g)$ vanishes along $\iota(U \times 0 \times g)$ with multiplicity $L(I)$;*
- *the map $\iota : U \times \Delta \times G_I \rightarrow \mathbb{P}\overline{A}_{\mathbf{g}, \mathbf{Z}, \mathbf{P}}^R$ is a degree 1 parametrization of a neighborhood of U in $\mathbb{P}\overline{A}_{\mathbf{g}, \mathbf{Z}, \mathbf{P}}^R$.*

The proof of Theorem 5 immediately follows from Lemma 3.27 because the vanishing order of $s_{j,i}$ along $\mathbb{P}\overline{A}_{\Gamma, I}$ is equal to

$$L(I) \cdot \text{Card}(G_I) = m(I).$$

Proof of Lemma 3.27. We prove the lemma in two steps: first we will prove the first three points of the lemma and then we will prove that ι is a parametrization of degree 1 of a neighborhood of U in $A_{\mathbf{g}, \mathbf{Z}, \mathbf{P}}^R$.

Proof of the first three points. For the sake of clarity we will successively prove the first three points at three levels of generality: first for a divisor graph with one edge, then for divisor graph with $R^1 = \{0\}$ and finally in full generality.

Divisor graph with one edge. For the moment we place ourselves in the simplest case: (Γ, I) is an admissible graph with two vertices, one at level 0 and one at level -1 . We suppose that there is only one edge with a twist given by $k > 0$. Let y_0 be a point of $\mathbb{P}A_{\Gamma, I}^{\text{gen}}$. Let U be an open neighborhood of y_0 in $\mathbb{P}A_{\Gamma, I}^{\text{gen}}$. A point y of U is given by

$$([C^0], [C^1], \overline{x}^0, \overline{x}^1, [\alpha^0]),$$

where C^0 and C^1 are the curves corresponding to the two vertices of the graph; \overline{x}^0 and \overline{x}^1 are their marked point sets; α^0 is a differential on the curve C^0 and $[\alpha^0]$ its equivalence class under the \mathbb{C}^* -action. More precisely, we denote by $\alpha^0(y)$ a nonvanishing section of the line bundle $\%_0(-1)$ over U . (Also recall that on C^1 the differential vanishes identically.)

The condition that $y \in A_{\Gamma, I}^{\text{gen}}$ implies that the curve C^1 carries a *unique* meromorphic differential α^1 with zeros and poles of prescribed multiplicities at the marked points, up to a scalar factor. Let $\alpha^1(y)$ be a nonvanishing section of the line bundle $\mathcal{N}_{\Gamma, I}$, i.e., a choice of the scalar factor for each point y .

At the neighborhood of the node the curves C^1 and C^0 have standard coordinates z and w such that $\alpha^0 = d(z^k)$ and $\alpha^1 = d(\frac{1}{w^k})$. The local coordinates z and w are unique up to the multiplication by a k^{th} root of unity. We fix one such choice in a uniform way over U . We define a family of curves $C(y, \epsilon)$ over $U \times \Delta$ by smoothing the node between C^0 and C^1 via the equation $zw = \epsilon$, where ϵ is the coordinate on the disc Δ and z, w are as above. The differentials α^0 and $\epsilon^k \alpha^1$ automatically glue together into a differential on $C(y, \epsilon)$.

The deformation that we have constructed does not depend on the choice of standard coordinates z and w . For instance, if we multiply z by a k^{th} root of unity ζ , the equation of the deformation becomes $zw = \zeta \epsilon$, which is isomorphic to the original deformation under a rotation of the disc Δ .

The section $s_{j,i}$ vanishes with multiplicity k along U : indeed we have explicitly

$$s_{j,i}(y, \epsilon) = \epsilon^k \cdot \alpha_1(y).$$

Divisor graph (Γ, I) with $R^1 = \{0\}$. We suppose now that the space R^1 is trivial (residues at the nodes between vertices of level 0 and -1 are equal to 0). A point y in U still determines

$$([C^0], [C^1], \bar{x}^0, \bar{x}^1, [\alpha^0], [\alpha^1])$$

where α^0 and α^1 are sections of $\mathcal{O}_0(-1)$ and $\mathcal{N}_{\Gamma, I}$ as in the previous paragraph.

Let e be an edge of Γ . We denote by k_e the positive integer equal to $|I(h)|$ for any of the two half-edges of e . Let z_e and w_e be a choice of standard coordinates in a neighborhood of the node corresponding to e : i.e. $\alpha^0 = d(z_e^{k_e})$ and $\alpha^1 = d(1/w_e^{k_e})$. This choice of standard coordinates being fixed for all edges, we choose, on top of that, ζ_e a k_e -th root of unity for each edge e .

We define a family of curves $C(y, \epsilon)$ over $U \times \Delta$ by smoothing the node corresponding to an edge e of Γ via the equation $z_e w_e = (\zeta_e \epsilon)^{L(I)/k_e}$ where ϵ is the coordinate on the disc Δ . The differentials defined by α^0 and by $\epsilon^{L(I)} \alpha_1$ automatically glue together into a differential on $C(y, \epsilon)$.

A multiplication of ϵ by a $L(I)$ -th root of unity ζ gives an isomorphic deformation. Thus two choices of roots $(\zeta_e)_{e \in \text{Edges}}$ and $(\zeta'_e)_{e \in \text{Edges}}$ gives isomorphic deformation if $\zeta'_e = \zeta^{L(I)/k_e} \zeta_e$ for all edges. Once again we get that the vanishing multiplicity of $s_{j,i}$ along U is $L(I)$.

General divisor graph (Γ, I) . We no longer impose restrictions on R^1 . We still define

$$([C^0], [C^1], \bar{x}^0, \bar{x}^1, \alpha^0, \alpha^1),$$

as above. Moreover we define the section r

$$r(y) = (r_e(y))_{e \in \text{Edges}},$$

where $r_e(y)$ is the residue of α_1 at the node of C^1 corresponding to the edge e . For every edge e , we fix a choice of standard coordinates of z_e and w_e in a neighborhood of the node corresponding to e , i.e., coordinates satisfying $\alpha^0 = d(z_e^{k_e})$ and $\alpha^1 = d(1/w_e^{k_e}) + \frac{r_e(y)dw_e}{w_e}$.

Using Proposition 2.16, we get a family of differentials $(\tilde{C}^0, \bar{x}^0, \tilde{\alpha}^0)$ parametrized by $U \times \Delta$ such that:

- when $\epsilon = 0$, we have $(C^0, \bar{x}^0, \alpha^0) = (\tilde{C}^0, \bar{x}^0, \tilde{\alpha}^0)$;
- the zeros of the differential which are not at the marked points corresponding to nodes are of fixed orders;
- the differential $\tilde{\alpha}^0$ has at most simple poles at the nodes of \tilde{C}^0 and the residue at the node corresponding to the edge e equals $-\epsilon^{L(I)} r_e(y)$;
- the vector of residues at the poles of $\tilde{\alpha}^0$ lies in R ;
- for each node corresponding to an edge e with a twist k_e , the family of differentials defined by $U \times \Delta$ is a standard deformation of $d(z_e^{k_e})$ (see Definition 2.12).

We use the fact that the family parametrized by $U \times \Delta$ is a standard deformation of $d(z_e^{k_e})$ to apply Proposition 2.13. At each node e the differential $\tilde{\alpha}_0$ can be written in the form $d(z_e^{k_e}) - \epsilon^{L(I)} r(u) \frac{dz_e}{z_e}$ in any annulus contained in a neighborhood of the node. Therefore we can still glue the two components together along this annulus with the identification $z_e w_e = \zeta_e \epsilon^{L(I)/k_e}$ for any choice of the k_e -th root of unity

ζ_e . The end of the proof is the same as for divisor graphs with trivial residue conditions.

Proof of the fourth point. Now we will prove that the map $\iota : U \times \Delta \times G_I \rightarrow \mathbb{P}\overline{A}_{\mathbf{g}, \mathbf{Z}, \mathbf{P}}^R$ is a degree 1 parametrization of a neighborhood of U in $\mathbb{P}\overline{A}_{\mathbf{g}, \mathbf{Z}, \mathbf{P}}^R$.

First we prove that the image $\iota(U \times \Delta \times G_I)$ covers *entirely* a neighborhood of U in $A_{\mathbf{g}, \mathbf{Z}, \mathbf{P}}^R$. Let $y_0 = (C = C_0 \cup C_1, \overline{x}_0, \overline{x}_1, \alpha_0)$ be a point in $A_{\Gamma, I}^{\text{gen}}$. Let $\tilde{\iota} : \Delta \rightarrow \overline{A}_{g, Z, P}$ be a family of differentials such that $\tilde{\iota}(0) = y_0$ and $\tilde{\iota}(\epsilon) \in A_{\mathbf{g}, \mathbf{Z}, \mathbf{P}}^R$ for $\epsilon \neq 0$. We denote by $\pi : \mathcal{C} \rightarrow \Delta$ the induced family of curves and by α the induced family of differentials on the fibers of $\mathcal{C} \rightarrow \Delta$.

Let e be a node of C with a twist of order k_e . Let γ_e be a simple loop in the curve C_0 around the node e . Let W_e be a neighborhood of γ_e in \mathcal{C} such that $W_e \cap \pi^{-1}(\epsilon)$ is an annulus for any ϵ small enough. Now, the differential α_0 is given by $d(z_e^{k_e})$ in a standard coordinate. Thus the differential $\alpha|_{\pi^{-1}(\epsilon)}$ is given by $d(z_e^{k_e}) + \phi(\epsilon, z_e)dz_e$ and we denote by $r_e(\epsilon)$ the integral of $\phi(\epsilon, z_e)dz_e$ along γ_e . We consider the differential $\alpha_e(\epsilon) = dz_e + \phi(\epsilon, z_e)dz_e - r_e(\epsilon)\frac{dz_e}{z_e}$. We fix a point p in the annulus $W_e \cap \pi^{-1}(\epsilon)$, the function $f : z \mapsto (\int_p^z \alpha_e)^{1/k_e}$ is uniquely determined for small values of ϵ . This determines a coordinate (that we will still denote z_e) such that $\alpha_0 = z_e^{k_e}dz_e - \varphi(\epsilon, z_e)\frac{dz_e}{z_e}$ with φ holomorphic and thus a standard deformation of α^0 . Proposition 2.13 implies that there exists a coordinate z_e on this annulus such that $\alpha|_{\pi^{-1}(\epsilon)} = d(z_e^{k_e}) + r_e(\epsilon)\frac{dz_e}{z_e}$.

We fix ϵ small enough so that the coordinates z_e are defined for all edges e . We cut the curve $\pi^{-1}(\epsilon)$ along simple loops contained in W_e . This gives two (possibly disconnected) curves with boundary C_0^{open} and C_1^{open} . We “plug” the holes of C_0^{open} with disks parametrized by the coordinate z_e and the holes of C_1^{open} with disks with coordinate $1/z_e$. This determines two curves $C_0(\epsilon)$ and $C_1(\epsilon)$. On both C_0 and C_1 , the local chart used to “plug” the holes allow us to define differentials $\alpha_0(\epsilon)$ and $\alpha_1(\epsilon)$.

The differential $\alpha_1(\epsilon)$ has a pole of order $k_e + 1$ at $w_e = 0$; thus $(C_1, \overline{x}_1, \alpha_1)(\epsilon)$ is an element of $A_{\mathbf{g}_1, \mathbf{Z}_1, \mathbf{P}_1}^{R^1}$. Now, at the level 0, we use Proposition 2.17: in a neighborhood of y_0 we can apply the retraction η . The point $\eta((C_0, \overline{x}_0, \alpha_0)(\epsilon))$ is a point of $A_{\mathbf{g}_0, \mathbf{Z}_0, \mathbf{P}_0}^{R^0}$. Therefore we define

$$y(\epsilon) = (\eta(C_0, \overline{x}_0, \alpha_0), (C_1, \overline{x}_1, \alpha_1))(\epsilon) \in A_{\Gamma, I}^{\text{gen}}.$$

For all ϵ in a neighborhood of 0, the point $\tilde{\iota}(\epsilon)$ lies in the deformation of $y(\epsilon)$ by the family ι restricted to $y(\epsilon) \times \Delta \times g$ for some $g \in G_I$ (in fact here $g = 1$ because of the choices of the parameters around y_0 that we have fixed).

To finish the proof of the fourth point, we need to prove that the parametrization is of degree 1. For this, we once again use the retraction η defined in Proposition 2.17. We have $\eta \circ \iota = \text{Id}_U$, thus we only need to prove that for all $y \in U$, the family ι restricted to $y \times \Delta \times G_I$ is of degree 1. We consider this family in the moduli space of curves, i.e let

$$\begin{aligned} \iota' : \Delta \times G_I &\rightarrow \overline{\mathcal{M}}_{\mathbf{g}, \mathbf{n}, \mathbf{m}} \\ \epsilon \times g &\mapsto p(\iota(y, \epsilon, G_I)). \end{aligned}$$

This family is of degree one. Indeed the stack $\overline{\mathcal{M}}_{\Gamma}$ is regularly imbedded in $\overline{\mathcal{M}}_{\mathbf{g}, \mathbf{n}, \mathbf{m}}$ and its normal bundle is the direct sum of the $T_h \otimes T_{h'}$ for all edges $e = (h, h')$ of

Γ . Thus the family ι' is given by the family:

$$\begin{aligned} \iota' : \Delta \times G_I &\rightarrow \bigoplus_{(h,h') \in \text{Edges}} T_h \otimes T_{h'} \\ (\epsilon, (\zeta_e)_{e \in \text{Edges}}) &\mapsto \left(\zeta_e \epsilon^{L(I)/k_e} \right)_{e \in \text{Edges}}, \end{aligned}$$

which is of degree 1. \square

Proof of Formula (2). We have seen that the space of differentials on an unstable component is a weighted projective space parametrized by

$$[w^{p-1} + a_1 w^{p-2} + \dots + a_{p-2} w] \frac{dw}{w},$$

where the weight of a_j is $\frac{j}{p-1}$. The fact that the order of the point x is $k_{j,i}$ is equivalent to the vanishing of the terms $a_{p-2}, \dots, a_{p-k_{j,i}-3}$. Therefore, the class of $[\mathbb{P}\overline{A}_{\mathbf{g}, \mathbf{Z}_{j,i}, \mathbf{P}}^R]$ is the closure of the vanishing locus $a_{p-k_{j,i}-2}$. Moreover we can easily check that $a_{p-k_{j,i}+1}^{p-1}$ is a global section of $\mathcal{O}_{\mathbb{P}}(-1)^{p-k_{j,i}+1}$. \square

3.6. Proof of Theorems 1, 2, and 3. We now have all ingredients to prove Theorem 4: for all $\mathbf{g}, \mathbf{Z}, \mathbf{P}$ (list of integers and vectors of integers) and R vector subspace of \mathcal{R} , the Poincaré-dual class of $[\mathbb{P}\overline{A}_{\mathbf{g}, \mathbf{Z}, \mathbf{P}}^R]$ in $H^*(\mathbb{P}\overline{\mathcal{H}}_{\mathbf{g}, \mathbf{Z}, \mathbf{P}}, \mathbb{Q})$ can be explicitly computed and is tautological (see Section 2.8).

Proof of Theorem 4. We prove Theorem 4 by induction on $|\mathbf{Z}| = \sum_{k \in \mathbf{Z}} k$.

Base of the induction: $|\mathbf{Z}| = 0$. If \mathbf{Z} is trivial then $A_{\mathbf{g}, \mathbf{Z}, \mathbf{P}}^R$ is dense in $\overline{\mathcal{H}}_{\mathbf{g}, \mathbf{n}, \mathbf{P}}$. Therefore

$$[\mathbb{P}A_{\mathbf{g}, \mathbf{Z}, \mathbf{P}}^R] = [\mathbb{P}\overline{\mathcal{H}}_{\mathbf{g}, \mathbf{n}, \mathbf{P}}^R] = \xi^{\dim(\mathcal{R}) - \dim(R)},$$

by Lemma 2.10.

Induction. Now we assume that $|\mathbf{Z}| > 0$. The induction Formulas (1) and (2) of Theorems 5 express the class $[\mathbb{P}\overline{A}_{\mathbf{g}, \mathbf{Z}, \mathbf{P}}^R]$ in terms of classes with smaller sum of the order of zeros. We only need to prove that the class $a_{\Gamma, I}$ is tautological for any $(\Gamma, I) \in \text{Div}(\mathbf{g}, \mathbf{Z}, \mathbf{P}, R)$.

The vectors of zeros \mathbf{Z}_0 and \mathbf{Z}_1 of the levels 0 and -1 satisfy $|\mathbf{Z}_i| < |\mathbf{Z}|$. Therefore the classes $[\mathbb{P}\overline{A}_{\mathbf{g}_1, \mathbf{Z}_1, \mathbf{P}_1}^{R^1}]$ and $[\mathbb{P}\overline{A}_{\mathbf{g}_0, \mathbf{Z}_0, \mathbf{P}_0}^{R^0}]$ can be computed and are tautological. Using Proposition 3.21, this implies that the class $a_{\Gamma, I}$ is tautological and can be computed. \square

Theorems 1, 2, and 3 stated in Section 1.5 are straightforward corollaries of Theorem 4.

Proof of Theorems 1, 2, and 3. Theorem 1 is the special case of Theorem 4 for a connected and stable curves. Theorem 3 is a consequence of 1 and Proposition 1.3 (the Segre class of the spaces of stable differential is tautological).

To prove Theorem 2, we recall that we denote by $\tilde{\pi}_n : \mathbb{P}\overline{\mathcal{M}}_{g,n} \rightarrow \mathbb{P}\overline{\mathcal{M}}_g$, the forgetful map of points. The bundle $\overline{\mathcal{H}}_{g,n}$ is the pull-back of $\overline{\mathcal{H}}_g$ by π_n , then $\xi \in H^*(\mathbb{P}\overline{\mathcal{H}}_{g,n}, \mathbb{Q})$ is the pull-back of $\xi \in H^*(\mathbb{P}\overline{\mathcal{H}}_g, \mathbb{Q})$. Therefore the push-forward of a tautological class of $RH^*(\mathbb{P}\overline{\mathcal{H}}_{g,n}, \mathbb{Q})$ by π_n is in $RH^*(\mathbb{P}\overline{\mathcal{H}}_g, \mathbb{Q})$ and can be explicitly computed.

If $Z = (k_1, \dots, k_n)$ is complete, the map $\tilde{\pi}_n$ restricted to $\mathbb{P}A_{g,Z}$ is finite of degree $\text{Aut}(Z)$ onto $\mathbb{P}\mathcal{H}[Z]$. We have

$$[\mathbb{P}\overline{\mathcal{H}}[Z]] = \frac{1}{\text{Aut}(K)} \cdot \tilde{\pi}_{n*} [\mathbb{P}\overline{A}_{g,Z}],$$

and the class $[\mathbb{P}\overline{\mathcal{H}}[Z]]$ is tautological and can be computed. \square

4. EXAMPLES OF COMPUTATION

We give two examples of computation: the first one is a computation in the projectivize Hodge bundle (we forget the marked points), the second is a computation in the moduli space of curves (we forget the differential).

4.1. The class $[\mathbb{P}\overline{\mathcal{H}}_g(3)]$. We consider here $g > 2$ and $Z = (3, 1, \dots, 1)$. We have seen in the introduction the computation of $[\mathbb{P}A_{g,(2)}]$. Therefore, in order to compute $[\mathbb{P}A_{g,(3)}]$ we need to list the divisor graphs contributing to $[\mathbb{P}\overline{A}_{g,(3)}] - (\xi + 3\psi_1)[\mathbb{P}\overline{A}_{g,(2)}]$.

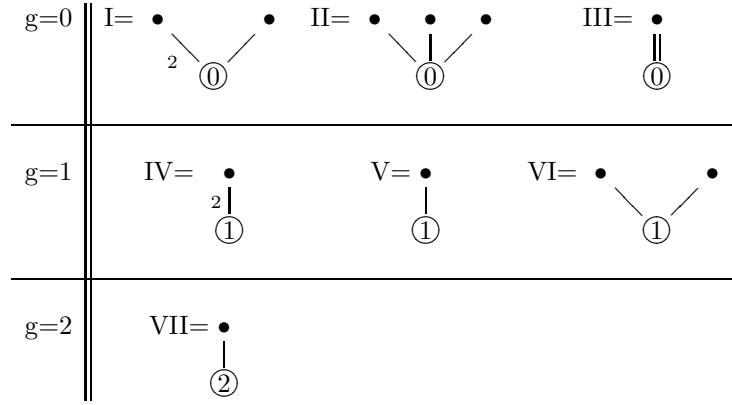


FIGURE 2. List of boundary terms in $[\mathbb{P}\overline{A}_{g,(3)}] - (\xi + 3\psi_1)[\mathbb{P}\overline{A}_{g,(2)}]$.

We have represented vertices of level -1 with their genera and the vertices of level 0 by bullets (the sum will run over all possible distribution of the genera of vertices of level 0). The marked point always belong to the unique vertex of level -1. The twists are represented by one number because the level structure already implies the sign of the twist on each half-edge. Finally we only represented the twists of absolute value greater than 1.

After push-forward by the forgetful map of the marked point, we get the following formula for the class $[\mathbb{P}\overline{\mathcal{H}}(3, 1, \dots, 1)] \in H^*(\mathbb{P}\overline{\mathcal{H}}_g, \mathbb{Q})$:

$$\begin{aligned} [\mathbb{P}\overline{\mathcal{H}}(3, 1, \dots, 1)] &= (12g - 12) \xi^2 + (11\kappa_1 - \delta - \delta_{sep} - 5 \textcircled{1} \text{---} \bullet) \xi \\ &\quad + \left(6\kappa_2 - \bullet^{\psi_e} \text{---} \bullet - 1/12 \textcircled{\textcircled{0}} \text{---} \bullet \right). \end{aligned}$$

Remark 4.1. For $g = 3$, we can compute $p_*[\mathbb{P}\overline{A}_{3,(3)}] \in H^0(\overline{\mathcal{M}}_3, \mathbb{Q}) \simeq \mathbb{Q}$, where p is the forgetful map of the differential. We get $p_*[\mathbb{P}\overline{A}_{g,(3)}] = 24$, the number of ordinary double points of a general quartic plane curve. In genus 3, we can also compute $p_*(\pi_*[\mathbb{P}\overline{A}_{3,(2,2)}]) = 2 \times 28$, i.e. two times the number of bitangents to a general quartic plane curve.

4.2. The class of $\overline{\mathcal{H}}_3(4)$. Here $g = 3$ and $\mu = 4$. We will compute the class $\overline{\mathcal{H}}_3(4) = \pi_*[\mathbb{P}\overline{A}_{3,(4)}] \in H^4(\overline{\mathcal{M}}_{3,1})$. We will not give the details of the computation however we have

$$\begin{aligned}
[\overline{\mathcal{H}}_3(4)] &= \lambda_2 - 10\psi_1\lambda_1 + 35\psi_1^2 - 5 \begin{array}{c} \textcircled{0}=\textcircled{2} \\ | \end{array} - \begin{array}{c} \textcircled{1}=\textcircled{1} \\ | \end{array} + 6 \begin{array}{c} \textcircled{1}-\textcircled{1}-\textcircled{1} \\ | \end{array} \\
&+ \begin{array}{c} \textcircled{1}-\textcircled{1}-\textcircled{1} \\ | \end{array} + 6 \begin{array}{c} \textcircled{1}-\textcircled{2} \\ | \lambda_1 \end{array} - 34 \begin{array}{c} \textcircled{1}-\textcircled{2} \\ | \psi_1 \end{array} - 11 \begin{array}{c} \textcircled{1}-\textcircled{2} \\ | \psi_e \end{array} \\
&+ \begin{array}{c} \textcircled{1}-\textcircled{2} \\ | \lambda_1 \end{array} - 10 \begin{array}{c} \textcircled{1}-\textcircled{2} \\ | \psi_1 \end{array} - \begin{array}{c} \textcircled{1}-\textcircled{2} \\ | \psi_e \end{array}
\end{aligned}$$

(the legs on the graphs stands for the only marked point). Felix Janda has compared this expression with the expression of Conjecture B. The two expressions agree modulo tautological relations (see Section 1.7 for presentation of the conjecture).

We recall that $\mathcal{H}_3(4)$ has two connected components (hyperelliptic and odd). In this case one can compute $[\overline{\mathcal{H}}_3(4)^{\text{hyp}}]$ by using the work of Faber and Pandharipande (see [11]). This way one can also compute $[\overline{\mathcal{H}}_3(4)^{\text{odd}}] = [\overline{\mathcal{H}}_3(4)] - [\overline{\mathcal{H}}_3(4)^{\text{hyp}}]$. In general, it is possible to compute the class of the hyperelliptic component but we do not know how to compute separately the classes of odd and even components for $g \geq 4$.

If we forget the marked point, then we get a class in $\text{Pic}(\overline{\mathcal{M}}_g) \otimes \mathbb{Q}$. The expression agrees with the formula of Scott Mullane (see [21]).

5. RELATIONS IN THE PICARD GROUP OF THE STRATA

We fix the notation for all the section. Let $g, n, m \geq 0$ such that $2g-2+n+m > 0$. Let $Z = (k_1, \dots, k_n)$ and $P = (p_1, \dots, p_n)$ be vectors of positive integers such that $|Z| - |P| = 2g - 2$. In this section we consider the space $\overline{\mathcal{H}}_g(Z - P) \subset \overline{\mathcal{M}}_{g,n+m}$ (see Section 1.5 for definitions). The purpose is to define several natural classes in $\text{Pic}(\overline{\mathcal{H}}_g(Z - P)) \otimes \mathbb{Q}$ and compute relations between these elements. Namely there are two types of classes which arise naturally:

- Divisors associated to admissible graphs (see Sections 3.2 and 3.3);
- Intersections of $\overline{\mathcal{H}}_g(Z - P)$ with the tautological classes of $A_1(\overline{\mathcal{M}}_{g,n})$.

5.1. Classes defined by admissible graphs. We consider the moduli space of stable differentials $\overline{\mathcal{H}}_{g,n,P}$ and the locus $\overline{A}_{g,Z,P} \subset \overline{\mathcal{H}}_{g,n,P}$. We have seen that a stratification of $\overline{A}_{g,Z,P}$ is obtained by considering admissible graphs (see Lemma 3.8). In this section, we describe the set of admissible graphs (Γ, I, ℓ) such that $p(\overline{A}_{\Gamma,I,\ell})$ is a divisor in $\overline{\mathcal{H}}_g(Z - P) = p(\overline{A}_{g,Z,P})$, where we recall that $p : \overline{\mathcal{H}}_{g,n,P} \rightarrow \overline{\mathcal{M}}_{g,n+m}$ is the forgetful map.

The map $p : \mathbb{P}A_{g,Z,P} \rightarrow \mathcal{H}_g(Z - P)$ is an isomorphism (see Lemma 2.18). Therefore an admissible graph (Γ, I, ℓ) which determines a divisor in $\overline{\mathcal{H}}_g(Z - P)$ needs necessarily to correspond to a divisor in $\overline{A}_{g,Z,P}$. We have seen that an admissible graph (Γ, I, ℓ) corresponds to a divisor of $\overline{A}_{g,Z,P}$ if and only it is of one of the three following types (see Section 3.3):

- (1) the admissible graph of depth 0 with one vertex and one edge;
- (2) an admissible graph of depth 0 with two vertices and one edge;

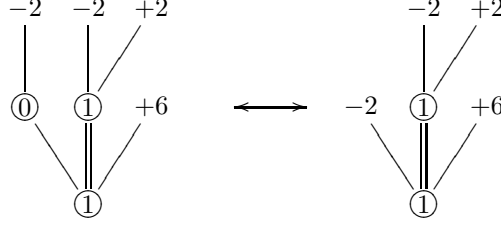


FIGURE 3. Example of the correspondance between admissible and stable graphs.

(3) a bicolored that satisfies the condition $(\star\star)$.

Proposition 5.1. *Let (Γ, I, ℓ) be an admissible graph. The locus $p(\mathbb{P}\overline{A}_{\Gamma, I, \ell})$ is a divisor of $\overline{\mathcal{H}}_g(Z - P)$ if and only if:*

- or (Γ, I, ℓ) is of the type 1 above ;
- or (Γ, I, ℓ) is a bicolored graph with one vertex of level -1 , one stable vertex of level 0 and possibly other semi-stable vertices of level 0 .

We call *irreducible divisor* the divisor of $\overline{\mathcal{H}}(Z - P)$ of the first type. We denote this divisor by D_0 (with reduced structure) and by δ_0 its class in $\text{Pic}(\overline{\mathcal{H}}_g(Z - P)) \otimes \mathbb{Q}$.

In the second case, the stabilization of the graph Γ determines a unique stable twisted graph of depth 1, (Γ', I') (we no longer write the level structure which is uniquely determined by I). Conversely given a twisted stable graph of depth 1 with two vertices, we can uniquely determine an admissible graph satisfying the condition of Proposition 5.1 by putting all the poles on the component of level -1 on unstable rational components of level 0 (see Example 5.3 below).

Definition 5.2. A *simple bicolored graph* is a twisted stable graphs of depth 1 with two vertices. We denote by $\text{SB}(Z, P)$ the set of simple bicolored graphs. If (Γ, I) is a simple bicolored graph, we denote by $D_{\Gamma, I}$ the corresponding divisor in $\overline{\mathcal{H}}_g(Z, P)$ (with the reduced structure) and by $a_{\Gamma, I}$ its class in $\text{Pic}(\overline{\mathcal{H}}_g(Z - P)) \otimes \mathbb{Q}$.

The class $i_* a_{\Gamma, I}$ (where i is the closed immersion of $p(\overline{A}_{\Gamma, I})$ in $\overline{\mathcal{M}}_{g, n+m}$) in the moduli space of curves is simply given by:

$$\zeta_{\Gamma*} ([\overline{\mathcal{H}}_{g_0}(Z_0 - P_0)], [\overline{\mathcal{H}}_{g_1}(Z_1 - P_1)]),$$

where g_0 and g_1 are the genera of the vertices of level 0 and -1 and the vectors Z_0 , P_0 , Z_1 and P_1 are the vectors encoding the orders of zeros and poles at the marked points and half-edges induced by Z, P and the twist I .

Example 5.3. We illustrate this correspondence between simple bicolored graphs and boundary divisors. We consider $g = 3$, $Z = (2, 6)$ and $P = (-2, -2)$ and the admissible graph (on this example we take the twists equal to 1 on all edges). On this example, the class $i_* a_{\Gamma, I}$ in the moduli space of curves will be given by

$$\zeta_{\Gamma*} ([\overline{\mathcal{H}}_1(+2, +0, +0, -2)], [\overline{\mathcal{H}}_1(+6, -2, -2, -2)]).$$

Proof of Proposition 5.1. Let (Γ, I) be an amissible graph of depth at most 1 with several stable components of level 0. Then the fiber of p over a generic point of $p(A_{g, Z, P})$ is of dimension greater than one. That is why divisors of type 2 are not

mapped to divisors while the map p restricted to D_0 is indeed of degree one onto its image.

Now we consider an admissible graph of depth 1 with at most one stable edge of level 0. Then the graph satisfies condition $(\star\star)$ if and only if it has one vertex of level -1 . \square

5.2. Class defined by residue conditions. We recall that \mathcal{R} is the vector space of residues, i.e. the subspace of \mathbb{C}^m defined by $\{(r_1, \dots, r_m) / r_1 + \dots + r_m = 0\}$. Let $1 \leq i < j \leq m$. Let $R_{i,j} \subset \mathcal{R}$ to be the vector subspace of \mathcal{R} defined by the equation $r_i = r_j$. The space $R_{i,j}$ is of codimension 1 in \mathcal{R} . We define the following class in the rational Picard group of $\overline{\mathcal{H}}_g(Z - P)$:

$$\delta_{ij}^{\text{res}} = p_*(\mathbb{P}\overline{A}_{g,Z,P}^{R_{i,j}}).$$

5.3. Classes defined by intersection. Let β be a tautological class in $\text{Pic}(\overline{\mathcal{M}}_{g,n+m}) \otimes \mathbb{Q}$. The class β determines a class in $\text{Pic}(\overline{\mathcal{H}}_g(Z - P)) \otimes \mathbb{Q}$ by taking $i^*\beta$ where i is the closed immersion of $\overline{\mathcal{H}}_g(Z - P)$ into $\overline{\mathcal{M}}_{g,n+m}$. If β is either λ_1, κ_1 or a ψ -class then we will denote by the same letter its pull-back to $\text{Pic}(\overline{\mathcal{H}}_g(Z - P)) \otimes \mathbb{Q}$ if the context is clear.

The last class that we will consider is the push-forward of the ξ -class that we denote:

$$\overline{\xi} = p_*(\xi \cdot [\mathbb{P}\overline{A}_{g,Z,P}]).$$

Notation 5.4. Let $1 \leq i < j \leq n + m$. We denote $\text{SB}(Z, P)_i$ (respectively $\text{SB}(Z, P)^i$) the set of simple bicolored graphs such that the leg corresponding to i is adjacent to the vertex of level -1 (respectively to the vertex of level 0). We denote $\text{SB}(Z, P)_i^j = \text{SB}(Z, P)_i \cap \text{SB}(Z, P)^j$.

Theorem 6. *The following relations holds in $\text{Pic}(\overline{\mathcal{H}}_g(Z - P)) \otimes \mathbb{Q}$:*

(1) *for all $1 \leq i \leq n$:*

$$\overline{\xi} + (k_i + 1)\psi_1 = \sum_{(\Gamma, I) \in \text{SB}(Z, P)_i} m(I)a_{\Gamma, I};$$

(2) *For all $1 \leq i, j \leq n$:*

$$(k_i + 1)\psi_i - (k_j + 1)\psi_j = \sum_{(\Gamma, I) \in \text{SB}(Z, P)_i^j} m(I)a_{\Gamma, I} - \sum_{(\Gamma, I) \in \text{SB}(Z, P)^i_j} m(I)a_{\Gamma, I};$$

(3) *if $m \geq 2$ then for all $1 \leq i < j \leq m$:*

$$\overline{\xi} = \delta_{i,j}^{\text{res}} + \sum_{(\Gamma, I) \in \text{SB}(Z, P)_i} m(I)a_{\Gamma, I} + \sum_{(\Gamma, I) \in \text{SB}(Z, P)^i_j} m(I)a_{\Gamma, I};$$

(4) *if $m = 0$ then*

$$\lambda_1 + \kappa_\mu \overline{\xi} = \delta + \sum_{(\Gamma, I) \in \text{SB}(Z, P)} 2\overline{m}(I, \Gamma)a_{\Gamma, I},$$

where δ is boundary divisor of $\overline{\mathcal{M}}_{g,n}$,

$$\kappa_\mu = \frac{1}{12} \sum_{i=1}^n \frac{k_i(k_i+2)}{k_i+1}$$

$$\text{and } \overline{m}(I, \Gamma) = \frac{m(I)}{12} \left(-m(I) + \sum_{i \rightarrow v^1} \frac{k_i(k_i+2)}{k_i+1} \right).$$

the second sums goes over all legs adjacent to the vertex of level -1 .

5.3.1. Relations (1) and (2) and Double Ramification cycles. The second relation of Theorem 6 is a direct consequence of the first one: we write $(k_i+1)\psi_i - (k_j+1)\psi_j = (\xi + (k_i+1)\psi_i) - (\xi + (k_j+1)\psi_j)$. However, we have chosed to write this relation in this form for two reasons:

- first because it involves only classes defined directly in the moduli space of curves;
- the second motivation is related to the Conjectures A and B. Indeed the classes $H_g(\mu)$ and $[\overline{\mathcal{H}}_g(\mu)]$ (see Section 1.7 for definitions) are supposed to be generalizations of the so-called Double Ramification cycles. In [3], the authors proved several identities between intersection of ψ -classes with the Double Ramification cycles. These relations are important in particular to construct the Double Ramification hierarchies.

Proof of Relation (1). It is a direct consequence of the induction formula (see Theorem 5). We consider Z_i , the vector obtained from Z by increasing the i -th entry by 1 and $R = \mathcal{R}$ (no residue condition), then we get:

$$(\xi + (k_i+1)\psi_i) \cdot [\mathbb{P}\overline{A}_{g,Z,P}] = [\mathbb{P}\overline{A}_{g,Z_j,P}] + \sum_{(\Gamma,I) \in \text{Bic}(g,Z,P)_i} m(I) a_{\Gamma,I}.$$

We remark that $|Z_j| - |P| > 2g - 2$ thus $[\mathbb{P}\overline{A}_{g,Z_j,P}] = 0$. Now we apply the push forward by p to this expression. In the sum of the right-hand side only the simple bicolored graphs will contribute and we indeed get

$$\xi + (k_i+1)\psi_1 = \sum_{(\Gamma,I) \in \text{SB}(Z,P)_i} m(I) a_{\Gamma,I}$$

□

5.3.2. Relation (3). To prove the third relation, we need a generalization of the induction formula:

Proposition 5.5. *The following equality holds in $H^*(\mathbb{P}\overline{\mathcal{H}}_{g,n,P}; \mathbb{Q})$*

$$[\mathbb{P}\overline{A}_{g,Z,P}^R] = \xi[\mathbb{P}\overline{A}_{g,Z,P}] - \sum_{(\Gamma,I) \in \text{Bic}(g,Z,P)_R} m(I) a_{\Gamma,I},$$

where $\text{Bic}(g,Z,P)_R$ is the set of bicolored graph such that the legs i and j are adjacent to distinct vertices.

Remark 5.6. We could have stated this proposition in a larger generality (unstable disconnected base) but it will not be useful here.

Proof. The proof is the same as the proof of Theorem 5. We consider the line bundle $\%_0(1) \simeq \%_0(-1)^\vee$ restricted to $\mathbb{P}\overline{A}_{g,Z,P}$ with its section

$$\begin{aligned} s : \%_0(-1) &\rightarrow \mathbb{C} \\ \alpha &\mapsto \text{res}_{x_i}(\alpha) - \text{res}_{x_j}(\alpha). \end{aligned}$$

The vanishing locus of the section s is the union of $\mathbb{P}\overline{A}_{g,Z,P}^R$ and of the divisors $\mathbb{P}\overline{A}_{\Gamma,I}$ for all $(\Gamma, I) \in \text{Bic}(g, Z, P)_R$.

Now the vanishing order of s along $\mathbb{P}\overline{A}_{g,Z,P}^R$ is 1 because the residue map is a submersion. The vanishing order of s along $\mathbb{P}\overline{A}_{\Gamma,I}$ is 1 because Lemma 3.27 remains valid if we replace the section s_{ij} by the section s and the set of graphs $\text{Div}(\mathbf{g}, \mathbf{Z}, \mathbf{P}, R)_{ij}$ by the set of graphs $\text{Div}(g, Z, P)_R$. \square

Proof of Relation (3). Now Relation (3) is a direct consequence of Proposition 5.5 by taking the push forward by the forgetful map p_* . \square

5.3.3. *Relation (4) and the work of Kontsevich and Zorich.*

Let $g \geq 2$. Let $Z = (k_1, k_2, \dots, k_n)$ be a partition of $2g - 2$. Let $\mathcal{M}(Z)$ be a connected component of $\mathbb{P}\overline{\mathcal{H}}_g(Z)$. There is a natural action of $PSL(2, \mathbb{R})$ on $\mathcal{H}_g(Z)$ and $\mathcal{M}(Z)$ is invariant under this action. Now we can consider the dynamic on $\mathcal{M}(Z)$ defined by the action of the diagonal $(e^{-t}, e^t)_{t \in \mathbb{R}}$. This allows to define a constant associated to this action: the sum of the Lyapunov exponents. Another invariant of the connected component c_Z is the Siegel-Veech constant c_Z (see [10]). The two constants are related by the relation

$$(3) \quad \mathcal{L}_Z = K_Z + c_Z,$$

where $K_Z = \frac{1}{12} \sum \frac{k_i(k_i+2)}{k_i+1}$.

Kontsevich proved the existence of a closed real 2-form β on $\mathcal{M}(Z)$, such that,

$$\mathcal{L}_Z = - \frac{\int_{\overline{\mathcal{M}(Z)}} \beta \wedge \lambda_1}{\int_{\overline{\mathcal{M}(Z)}} \beta \wedge \xi}.$$

Here λ_1 and ξ are considered as elements in $H^2(\mathbb{P}\overline{\mathcal{H}}_g(Z), \mathbb{R})$. Relation (3) comes from the following equality in $H^2(\mathbb{P}\overline{\mathcal{H}}_g(Z), \mathbb{R})$:

$$\lambda_1 = K_Z(-\xi) + \delta,$$

and from:

$$c_Z = - \frac{\int_{\overline{\mathcal{M}(Z)}} \beta \wedge \delta}{\int_{\overline{\mathcal{M}(Z)}} \beta \wedge \xi}.$$

The class δ lies in the boundary of $\mathbb{P}\overline{\mathcal{H}}_g(Z)$. The fourth relation of Theorem 6 is the explicit computation of the boundary class in $\text{Pic}(\overline{\mathcal{H}}_g(Z)) \otimes \mathbb{Q}$.

Remark 5.7. Here we compute the boundary terms in the compactification of the strata inside the moduli space of curves. Another formula can be obtained in the Hodge bundle. The two are very similar but we prefer to state the formula in this form to complete our study of the Picard group of $\overline{\mathcal{H}}_g(Z)$.

Proof of Relation (4). We consider $g > 0$, $P = 0$ and $Z = (k_1, \dots, k_n)$ a partition of $2g - 2$. Let Z' be the vector equal to $(k_1, \dots, k_n, 0)$. If $\pi : \overline{\mathcal{M}}_{g,n+1} \rightarrow \overline{\mathcal{M}}_{g,n}$ is

the forgetful map of the last marked point, then we have $\overline{A}_{g,Z'} = \pi^{-1}(\overline{A}_{g,Z})$. We use the induction formula to obtain the relation:

$$(\xi + \psi_{n+1})[\mathbb{P}A_{g,Z'}] = 0 + \sum_{\text{Bic}(g,Z)_{n+1}} m(I) a_{\gamma,I}$$

We multiply this formula by ψ_{n+1} to get

$$(4) \quad \xi \psi_{n+1}[\mathbb{P}A_{g,Z'}] + \psi_{n+1}^2[\mathbb{P}A_{g,Z'}] = \sum_{\text{Bic}(g,Z')_{n+1}} m(I) \psi_{n+1} a_{\Gamma,I}.$$

Now we apply $(p_*) \circ (\pi_*)$ to this formula (we forget the last point and then the differential). We study each term separately.

Contribution of $\xi \psi_{n+1}[\mathbb{P}A_{g,Z'}]$. The classes ξ and $[\mathbb{P}A_{g,Z'}]$ are pull back by π thus

$$\begin{aligned} p_* (\pi_* (\psi_{n+1} \xi [\mathbb{P}A_{g,Z'}])) &= p_* (\pi_* (\psi_{n+1}) \xi [\mathbb{P}A_{g,Z}]) \\ &= \kappa_0 p_* (\xi [\mathbb{P}A_{g,Z}]) \\ &= (2g - 2 + n) \bar{\xi} \end{aligned}$$

by the projection formula.

Contribution of $\psi_{n+1}^2[\mathbb{P}A_{g,Z'}]$. Still by the projection formula we have:

$$\begin{aligned} p_* (\pi_* (\psi_{n+1}^2 [\mathbb{P}A_{g,Z'}])) &= p_* (\pi_* (\psi_{n+1}^2) [\mathbb{P}A_{g,Z}]) \\ &= \kappa_1 \\ &= 12\lambda_1 - \delta + \sum_{i=1}^n \psi_i. \end{aligned}$$

Now we use the first relation to write:

$$\sum_{i=1}^n \psi_i = - \left(\sum_{i=1}^n \frac{1}{k_i + 1} \right) \bar{\xi} + \sum_{i=1}^n \left(\sum_{(\Gamma,I) \in \text{BS}(g,Z)_i} \frac{m(I)}{k_i + 1} a_{\Gamma,I} \right).$$

Contribution of $\sum_{\text{Bic}(g,Z')_{n+1}} m(I) \psi_{n+1} a_{\Gamma,I}$. Let (Γ, I) be a bicolored graph in $\text{Bic}(g, Z')_{n+1}$. There are two possible configurations:

- the point $n+1$ belongs to a rational components with 3 special points. In which case $\psi_{n+1} a_{\Gamma,I} = 0$;
- the point $n+1$ is carried by general vertex of level -1 which is not contracted after the forgetful map.

In the second case, we denote by (Γ', I') the twisted graph obtained after forgetting the marked point. We get:

$$\pi_* (\psi_{n+1} a_{\Gamma,I}) = (2g_{\Gamma',I',1} - 2 + n_{\Gamma',I',1}) a_{\Gamma',I'},$$

where $g_{\Gamma,1}$ and $n_{\Gamma,1}$ denote the genus and valency of the vertex of level -1 . Thus

$$(p_* \circ \pi_*) \sum_{\text{Bic}(g,Z')_{n+1}} m(I) \psi_{n+1} a_{\Gamma,I} = \sum_{(\Gamma,I) \in \text{BS}(g,Z)} m(I) (2g_{\Gamma',I',1} - 2 + n_{\Gamma',I',1}) a_{\Gamma,I}.$$

We obtain Relation (4) by replacing all the terms in Equation (4) by their expressions in terms of simple bicolored graphs. \square

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